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Efficient low rank matrix recovery with flexible group sparse regularization

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In this paper, we present a novel approach to the low rank matrix recovery (LRMR) problem by casting it as a group sparsity problem. Specifically, we propose a flexible group sparse regularizer (FLGSR) that can group any number of matrix columns as a unit, whereas existing methods group each column as a unit. We prove the equivalence between the matrix rank and the FLGSR under some mild conditions, and show that the LRMR problem with either of them has the same global minimizers. We also establish the equivalence between the relaxed and the penalty formulations of the LRMR problem with FLGSR. We then propose an inexact restarted augmented Lagrangian method, which solves each subproblem by an extrapolated linearized alternating minimization method. We analyse the convergence of our method. Remarkably, our method linearizes each group of the variable separately and uses the information of the previous groups to solve the current group within the same iteration step. This strategy enables our algorithm to achieve fast convergence and high performance, which are further improved by the restart technique. Finally, we conduct numerical experiments on both grayscale images and high altitude aerial images to confirm the superiority of the proposed FLGSR and algorithm.

Keywords: low rank matrix recovery; flexible group sparse regularizer; capped folded concave function.

1. Introduction

The recovery of an unknown low rank matrix $C \in \mathbb{R}^{m\times n}$ from very limited information has arisen in many applications, such as optimal control Fazel *et al.* (2001, 2004), image classification Cabral *et al.* (2015); Luo *et al.* (2015), multi-task learning Abernethy *et al.* (2006); Amit *et al.* (2007), image inpainting Komodakis (2006); Ma *et al.* (2017); Yu & Zhang (2022) and so on. The problem is formulated as the following low rank matrix recovery (LRMR) problem:

$$\min_{C\in\mathbb{R}^{m\times n}} \operatorname{rank}(C), \quad \text{s.t.} \quad \|\mathscr{A}(C)-\boldsymbol{b}\|_2 \le \sigma, \tag{1.1}$$

where C is the decision variable, and the linear transformation $\mathscr{A}: \mathbb{R}^{m\times n} \to \mathbb{R}^p$ and vector $\boldsymbol{b} \in \mathbb{R}^p$ are given.

Problem (1.1) is NP-hard because of the combinatorial property of the rank function. To solve problem (1.1), the rank function is relaxed by various spectral functions, such as the nuclear norm, the truncated nuclear norm, the Schatten-q quasi-norm, etc. Under mild conditions, the low rank matrix can be exactly recovered from most sampled entries by minimizing the nuclear norm of the matrix

Candès & Recht (2009). Therefore, the nuclear norm minimization has been widely studied for LRMR problem Candès & Recht (2009); Cai *et al.* (2010); Recht *et al.* (2010); Zhang *et al.* (2010); Chen *et al.* (2011); Ma *et al.* (2011), which leads to a convex optimization problem. Numerical methods for nuclear norm minimization problem have strong theoretical guarantees under some conditions, which cannot be satisfied in some practical applications Candès & Recht (2009); Candes & Tao (2010). In other words, the nuclear norm is not the best approximation of the rank function. In this regard, the nonconvex relaxations are tighter than the nuclear norm relaxation to the rank function. Some popular nonconvex relaxation include truncated nuclear norm Lee & Lam (2016); Dong *et al.* (2018); Su *et al.* (2019), capped-l_1 function Yu & Zhang (2022), (truncated) l_{1-2} metric Ma *et al.* (2017); Ge & Li (2021); An *et al.* (2023) and Schatten-q quasi-norm Marjanovic & Solo (2012); Nie *et al.* (2012); Liu *et al.* (2018). Note that all of these methods have to compute singular value decompositions (SVD) in each iteration, which leads to high computational cost. To cut down the computation cost, Keshavan *et al.* (2010), van den Eriksson & Hengel (2012), Wen *et al.* (2012) and Xu *et al.* (2012) adopted a low rank matrix factorization to preserve the low rank structure of matrix such that problem (1.1) is relaxed as

$$\min_{X\in\mathbb{R}^{m\times r}, Y\in\mathbb{R}^{n\times r}, C\in\mathbb{R}^{m\times n}} \frac{1}{2}\left\|XY^T - C\right\|_F^2, \quad \text{s.t.} \quad \|\mathscr{A}(C) - \boldsymbol{b}\|_2 \le \sigma, \tag{1.2}$$

where r is a preestimated matrix rank. For problem (1.2), the running time of numerical methods is cut down dramatically, while the performance of the methods based on matrix factorization is not satisfactory Fan *et al.* (2019); Yao *et al.* (2019).

Recently, group sparse regularizer as a surrogate for the matrix rank has attracted more and more interest as it scales well to large-scale problems. For example, Fan *et al*. (Fan *et al.* (2019)) proposed a new class of factor group-sparse regularizers (FGSR) as a surrogate for the matrix rank. To solve the matrix recovery problem associated with the proposed FGSR, they used the alternating direction method of multipliers (ADMM) with linearization. Jia *et al*. (Jia *et al.* (2021)) proposed a generalized unitarily invariant gauge (GUIG) function for LRMR problem and solved it using the accelerated block prox-linear algorithm. Although the above group sparse regularizer based matrix recovery models have achieved satisfactory performance, they still suffer from the following limitations: (1) They treated each column of the matrix as a group and designed their algorithms to linearize the entire matrix instead of each column, which would otherwise dramatically increase the running time as the matrix size grows. However, a drawback of this approach is that they could not use the information of the previous groups to solve the current group in the same iteration step. This resulted in poor solution quality. (2) Although they established the relationship between the proposed group sparse regularizer and the matrix rank, they lacked the equivalence analysis of the related problem.

In this paper, we introduce an FLGSR for the LRMR problem. The proposed FLGSR is partly based on the group sparse regularizer studied in Fan *et al.* (2019); Jia *et al.* (2021), but generalizes it to the flexible group setting. Our method, based on FLGSR, outperforms spectral functions, matrix factorization and previous group sparse regularizers in performance and efficiency for large-scale problems. It avoids computing SVD, estimating matrix rank, and allows flexible column grouping. In summary, our contributions can be summarized as follows:

(1) We prove that the matrix rank can be formulated equivalently as an FLGSR under some simple conditions. Our FLGSR model is more flexible than the FGSR model proposed in Fan *et al.* (2019) and the GUIG model proposed in Jia *et al.* (2021) because it can group an arbitrary number of matrix columns as a unit, whereas FGSR and GUIG group each column as a unit.

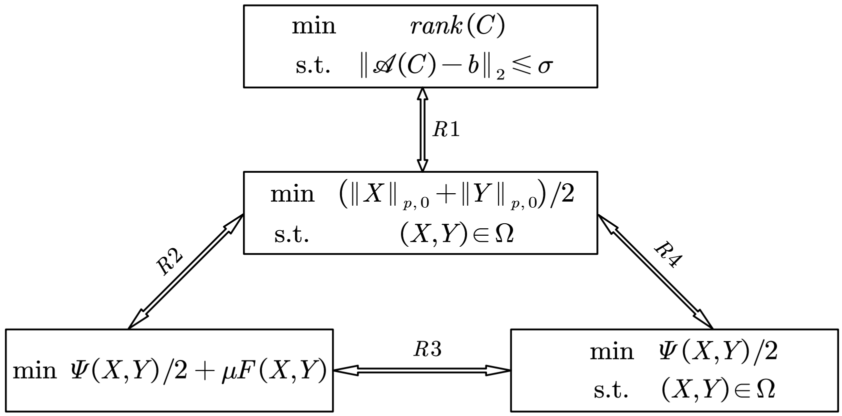

FIG. 1. The relationships of global minimizers between problems (1.1), (3.1), (3.2) and (3.3). R1 [Theorem 3], R2 [Theorem 6], R3 [Theorem 5], R4 [Theorem 4].

(2) We show that the LRMR problem based on matrix rank and the one based on FLGSR have the same global minimizers. Moreover, we prove the equivalence between the LRMR problem based on FLGSR with ℓ_0-norm and its relaxed version, as well as the equivalence between the relaxed version and the corresponding penalty problem. Their links are summarized in Fig. 1.

(3) We propose an inexact restarted augmented Lagrangian method, whose subproblem at each iteration is solved by an extrapolated linearized alternating minimization method. We also provide the convergence analysis of our method. In the update subproblem, we linearize each group, so that we can utilize the information from the groups that have been iterated before when we iterate the current group. This strategy enables our algorithm to achieve fast convergence and high performance, which are further enhanced by the restarted technique.

Notation. We introduce some notations that will be used throughout this paper. We denote by $[n]$ the set $\{1, 2, \ldots, n\}$, where n is a positive integer. Scalars, vectors and matrices are denoted by lowercase letters (e.g., x), boldface lowercase letters (e.g., $\boldsymbol{x}$) and uppercase letters (e.g., X), respectively. The notation $\|\boldsymbol{x}\|_2 := \sqrt{\sum_i x_i^2}$ denotes the ℓ_2-norm of a vector $\boldsymbol{x}$. For a matrix X, we denote by $\sigma(X) := (\sigma^1(X), \sigma^2(X), \ldots)$ the singular value vector of X arranged in a nonincreasing order. The Frobenius norm, the ℓ_p-norm, the nuclear norm and the spectral norm of a matrix X are defined as $\|X\|_F := \sqrt{\sum_i \sum_j X_{ij}^2}$, $\|X\|_p := \sqrt[p]{\sum_i \sum_j |X_{ij}|^p}$, $\|X\|_* := \sum_i \sigma^i(X)$ and $\|X\| := \sigma^1(X)$, respectively. Let X be partitioned into s disjoint groups as $X := [X_1, \ldots, X_s] \in \mathbb{R}^{m\times n}$ such that $X_i := X(:, \sum_{l=0}^{i-1} n_l + 1 : \sum_{l=0}^{i} n_l) \in \mathbb{R}^{m\times n_i}$ for all $i \in [s]$ (adopting the convenience that $n_0 = 0$). Here, $n_1, \ldots, n_s$ are positive integers satisfying $\sum_{i=1}^{s} n_i = n$. Then we denote the $\ell_{p,0}$-norm of X as $\|X\|_{p,0} := \sum_{i=1}^{s} n_i \|X_i\|_p^0$ (adopting the convenience that $0^0 = 0$) with $p \geq 1$. We also denote the group support set of X as

$$\Gamma(X) := \left\{ i \mid \left\|X_i\right\|_p \neq 0,\ i = 1, \ldots, s \right\} = \Gamma_1(X) \cup \Gamma_2(X),$$

$$\Gamma_1(X) := \left\{ i \mid \left\| X_i \right\|_p < \nu, i \in \Gamma(X) \right\}, \; \Gamma_2(X) := \left\{ i \mid \left\| X_i \right\|_p \geq \nu, i \in \Gamma(X) \right\}.$$

The p-distance from X to a closed set $\mathbb{S} \subseteq \mathbb{R}^n$ is defined by $\text{dist}_p(X, \mathbb{S}) := \inf\{\|X - Y\|_p : Y \in \mathbb{S}\}$.

2. Flexible group sparse regularizer

Following Fan *et al.* (2019), the rank of a matrix $C \in \mathbb{R}^{m\times n}$ can be written as

$$\text{rank}(C) = \min_{C=XY^T} \text{nnzc}(X) = \min_{C=XY^T} \text{nnzc}(Y) = \min_{C=XY^T} \frac{1}{2}(\text{nnzc}(X) + \text{nnzc}(Y)),$$

where nnzc $(\cdot)$ denote the number of nonzero columns of the matrix. However, directly solving the model corresponding to the above decomposition is difficult due to its nonsmoothness. Therefore, Fan *et al.* (2019) proposed the following FGSR:

$$\text{FGSR}_{1/2}(C) := \frac{1}{2} \min_{C=XY^T} \left(\|X\|_{2,1} + \|Y\|_{2,1} \right),$$

$$\text{FGSR}_{2/3}(C) := \frac{2}{3\alpha^{1/3}} \min_{C=XY^T} \left(\|X\|_{2,1} + \frac{\alpha}{2} \|Y\|_F^2 \right),$$

where $\alpha > 0$, $X \in \mathbb{R}^{m\times d}$ and $Y \in \mathbb{R}^{n\times d}$ with rank$(C) \leq d \leq \min\{m, n\}$. $\|X\|_{2,1} := \sum_{j=1}^{d} \|X(:,j)\|_2$ and the same to $\|Y\|_{2,1}$. Furthermore, Jia *et al.* (2021) gave a GUIG function:

$$\text{GUIG}_g(C) = \min_{C=XY^T} \sum_{j=1}^{d} g_1\left(\|X(:,j)\|_2\right) + \sum_{j=1}^{d} g_2\left(\|Y(:,j)\|_2\right),$$

where $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ are two functions.

We observe that the grouping methods of FGSR and GUIG are not flexible enough, since they both consider each column of the matrix as a group. This has the following disadvantages:

- When the algorithm applies alternating minimization to the whole matrix, it updates X^k, Y^k, X^{k+1}, Y^{k+1}, etc., sequentially. However, this implies that it cannot utilize the information from $X^{k+1}(:,l), l < j$ when updating $X^{k+1}(:,j)$, or the information from $Y^{k+1}(:,l), l < j$ when updating $Y^{k+1}(:,j)$. This results in poor effectiveness.
- When the algorithm applies alternating minimization to each group of the matrix, it updates $X^k(:,1)$, $\ldots$, $X^k(:,d)$, $Y^k(:,1)$, $\ldots$, $Y^k(:,d)$, $X^{k+1}(:,1)$, $\ldots$, $X^{k+1}(:,d)$, etc., in turn. The computational cost grows significantly with the number of columns of the matrix.

Therefore, in order to balance the efficiency and effectiveness, we propose a flexible grouping scheme, and extend the group ℓ_2-norm to a more general group ℓ_p-norm with $p \geq 1$, named FLGSR, defined as follows:

Definition 1. Let $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ be functions. The FLGSR of a matrix $C \in \mathbb{R}^{m\times n}$, denoted by $G_p^{\phi_1,\phi_2}(\cdot) : \mathbb{R}^{m\times n} \to \mathbb{R}$, is defined as follows:

$$G_p^{\phi_1,\phi_2}(C) = \min_{C=XY^T} \frac{1}{2}\sum_{i=1}^{s} n_i\left(\phi_1\left(\|X_i\|_p\right) + \phi_2\left(\|Y_i\|_p\right)\right),$$

where $\sum_{i=1}^{s} n_i = n$, $X := [X_1,\dots,X_s] \in \mathbb{R}^{m\times n}$ with $X_i \in \mathbb{R}^{m\times n_i}$ and $Y := [Y_1,\dots,Y_s] \in \mathbb{R}^{n\times n}$ with $Y_i \in \mathbb{R}^{n\times n_i}$ for all $i \in [s]$.

Remark 1. Thanks to the flexible grouping in FLGSR, we can select large n_i values when n is large, making s much smaller than n. This enables us to design an algorithm that applies alternating minimization to each group of the matrix, improving both computational efficiency and effectiveness. Please see Subsection 5.1.1 for detailed comparison of different number of groups.

For simplicity, we denote $G_p^{\phi,\phi}(C)$ by $G_p^{\phi}(C)$. In what follows, we present two theorems that reveal the relationships between $G_p^{|\cdot|^0}(C)$ and $\operatorname{rank}(C)$, and between $G_p^{\phi_1,\phi_2}(C)$ and the nuclear p-norm of matrix C Friedland & Lim (2017), respectively.

Theorem 1. Let $C = [C_1,\dots,C_s] \in \mathbb{R}^{m\times n}$ be a matrix of rank r, where $C_i \in \mathbb{R}^{m\times n_i}$ with $\sum_{i=1}^{s} n_i = n$. If there are $n_{i_1},\dots,n_{i_p} \in \left\{n_1,\dots,n_s\right\}$ such that $\sum_{j=1}^{p} n_{i_j} = r$, then $G_p^{|\cdot|^0}(C) = \operatorname{rank}(C)$.

Proof. On the one hand, from the definition of $\|\cdot\|_{p,0}$, for any matrices $X := \left[X_1,\dots,X_s\right] \in \mathbb{R}^{m\times n}$ and $Y := \left[Y_1,\dots,Y_s\right] \in \mathbb{R}^{n\times n}$ that satisfy $C = XY^T$, we obtain that

$$\begin{aligned}\operatorname{rank}(C) &\le \frac{1}{2}\left(\operatorname{rank}(X) + \operatorname{rank}(Y)\right) = \frac{1}{2}\operatorname{rank}\left(\left[X_1,\dots,X_s\right]\right) + \frac{1}{2}\operatorname{rank}\left(\left[Y_1,\dots,Y_s\right]\right)\\ &\le \frac{1}{2}\sum_{i=1}^{s}\left(\operatorname{rank}\left(X_i\right) + \operatorname{rank}\left(Y_i\right)\right) \le \frac{1}{2}\sum_{i=1}^{s} n_i\left(\left\|X_i\right\|_p^0 + \left\|Y_i\right\|_p^0\right) = \frac{1}{2}\left(\|X\|_{p,0} + \|Y\|_{p,0}\right).\end{aligned}$$

On the other hand, it is clear that there exists two column full rank matrices $\bar{X} \in \mathbb{R}^{m\times r}$ and $\bar{Y} \in \mathbb{R}^{n\times r}$ such that $C = \bar{X}\bar{Y}^T$. By the given conditions, we know that there exist $n_{i_1},\dots,n_{i_p} \in \left\{n_1,\dots,n_s\right\}$ such that $\sum_{j=1}^{p} n_{i_j} = r$. Without loss of generality, we suppose $n_{i_1} = n_1,\dots,n_{i_p} = n_p$. Let $\tilde{X} = [\bar{X}, 0] \in \mathbb{R}^{m\times n}$ and $\tilde{Y} = [\bar{Y}, 0] \in \mathbb{R}^{n\times n}$, then $C = \tilde{X}\tilde{Y}^T$ and $\operatorname{rank}(\tilde{X}) = \|\tilde{X}\|_{p,0}$, $\operatorname{rank}(\tilde{Y}) = \|\tilde{Y}\|_{p,0}$.

From the above two aspects, we can conclude that

$$\operatorname{rank}(C) = \min_{C=XY^T} \frac{1}{2}\left(\|X\|_{p,0} + \|Y\|_{p,0}\right) = G_p^{|\cdot|^0}(C).$$

The proof is completed. □

Definition 2. The generalized flexible nuclear p-norm of a matrix $C \in \mathbb{R}^{m\times n}$ is

$$\|C\|_{*,p}^{\phi} = \inf\left\{\sum_{i=1}^{s} n_i\phi\left(\left|\lambda_i\right|\right) : C = \sum_{i=1}^{s} \lambda_i U_i V_i^T, \left\|U_i\right\|_p = \left\|V_i\right\|_p = 1\right\} \tag{2.1}$$

for any $p \ge 1$. Here, $U_i \in \mathbb{R}^{m\times n_i}$, $V_i \in \mathbb{R}^{n\times n_i}$ for $i \in [s]$ and $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$.

If we set $\phi(x) = x$ and $n_i = 1$ for $i \in [s]$, then (2.1) reduces to nuclear p-norm Friedland & Lim (2017):

$$\|C\|_{*,p} = \inf\left\{\sum_{i=1}^{\min\{m,n\}} \left|\lambda_i\right| : C = \sum_{i=1}^{\min\{m,n\}} \lambda_i \boldsymbol{u_i}\boldsymbol{v_i}^T, \|\boldsymbol{u_i}\|_p = \|\boldsymbol{v_i}\|_p = 1\right\}.$$

THEOREM 2. For any lower bounded function ϕ, if there exist functions ϕ_1 and ϕ_2 such that $\phi(z) = \frac{1}{2}\min_{z=xy} \phi_1(x) + \phi_2(y)$, then $\|C\|_{*,p}^{\phi} = G_p^{\phi_1,\phi_2}(C)$.

Proof. Let $C = \sum_{i=1}^{s} \lambda_i U_i V_i^T$ be an arbitrary decomposition of the matrix C, where $\|U_i\|_p = \|V_i\|_p = 1$ for $i \in [s]$. Given $\lambda_i = \lambda_i^1\lambda_i^2$ and $\phi(|\lambda_i|) = \frac{1}{2}\min_{|\lambda_i|=|\lambda_i^1||\lambda_i^2|} \phi_1(|\lambda_i^1|) + \phi_2(|\lambda_i^2|)$ from the known conditions, we have

$$\inf_{\lambda_i} \phi\left(\left|\lambda_i\right|\right) = \frac{1}{2}\inf_{\lambda_i^1,\lambda_i^2} \phi_1\left(\left|\lambda_i^1\right|\right) + \phi_2\left(\left|\lambda_i^2\right|\right) \tag{2.2}$$

because the infimum is equal to the minimum for bounded functions.

By reformulating the matrix C, we obtain

$$C = \sum_{i=1}^{s} \lambda_i U_i V_i^T = \sum_{i=1}^{s} \underbrace{\lambda_i^1 U_i}\underbrace{(\lambda_i^2 V_i)}^T.$$

Let $X_i = \lambda_i^1 U_i$ and $Y_i = \lambda_i^2 V_i$. Then from $\|U_i\|_p = \|V_i\|_p = 1$, we have $|\lambda_i^1| = \|X_i\|_p$ and $|\lambda_i^2| = \|Y_i\|_p$. Combining this with (2.2) gives

$$\inf_{\lambda_i} \phi\left(\left|\lambda_i\right|\right) = \frac{1}{2}\inf_{X_i,Y_i} \phi_1\left(\left\|X_i\right\|_p\right) + \phi_2\left(\left\|Y_i\right\|_p\right). \tag{2.3}$$

Summing up the above inequality over $i = 1, \ldots, s$, it gives

$$\inf_{\boldsymbol{\lambda}} \sum_{i=1}^{s} n_i \phi\left(\left|\lambda_i\right|\right) = \min_{C=XY^T} \frac{1}{2}\sum_{i=1}^{s} n_i \left(\phi_1\left(\left\|X_i\right\|_p\right) + \phi_2\left(\left\|Y_i\right\|_p\right)\right).$$

The proof is completed. □

REMARK 2. According to Theorems 1 and 2, the matrix rank and the generalized flexible nuclear p-norm $\|C\|_{*,p}^{\phi}$ can be equivalently formulated as our proposed FLGSR functions $G_p^{|\cdot|^0}(C)$ and $G_p^{\phi_1,\phi_2}(C)$, respectively, under some mild conditions. Based on our proposed FLGSR, we do not need to calculate the SVD of the matrix, which is a costly operation, unlike the spectral function of the matrix. It also does not need to estimate the rank of the matrix beforehand, which is difficult and can affect the model performance if it is too high or too low, unlike the matrix decomposition. Moreover, our grouping is more flexible than other group sparse regularizers, allowing us to design a faster algorithm with better results.

3. Efficient LRMR with FLGSR: equivalence analysis

In this section, we propose a novel FLGSR model based on matrix factorization for the LRMR problem (1.1), as follows:

$$G_p^{|\cdot|^0}\left(XY^T\right) = \min_{X\in\mathbb{R}^{m\times n}, Y\in\mathbb{R}^{n\times n}} \frac{1}{2}\left(\|X\|_{p,0} + \|Y\|_{p,0}\right), \quad \text{s.t.} \quad \left\|\mathscr{A}\left(XY^T\right) - \boldsymbol{b}\right\|_2 \le \sigma. \tag{3.1}$$

Then, we consider the following relaxation problem of problem (3.1):

$$\begin{aligned} G_p^{\phi}\left(XY^T\right) &= \min_{X\in\mathbb{R}^{m\times n}, Y\in\mathbb{R}^{n\times n}} \frac{1}{2}\sum_{i=1}^{s} n_i\left(\phi\left(\left\|X_i\right\|_p\right) + \phi\left(\left\|Y_i\right\|_p\right)\right) \\ \text{s.t.} &\quad \left\|\mathscr{A}\left(XY^T\right) - \boldsymbol{b}\right\|_2 \le \sigma, \end{aligned} \tag{3.2}$$

and its penalty problem:

$$\min_{X\in\mathbb{R}^{m\times n}, Y\in\mathbb{R}^{n\times n}} \frac{1}{2}\sum_{i=1}^{s} n_i\left(\phi\left(\left\|X_i\right\|_p\right) + \phi\left(\left\|Y_i\right\|_p\right)\right) + \mu \max\left\{\left\|\mathscr{A}\left(XY^T\right) - \boldsymbol{b}\right\|_2^2 - \sigma^2, 0\right\}. \tag{3.3}$$

Here, function $\phi(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a capped folded concave function that satisfies the following two conditions with a fixed parameter $\nu > 0$:

(1) ϕ is continuous, increasing and concave in $[0, \infty)$ with $\phi(0) = 0$;

(2) there is a $\nu > 0$ such that ϕ is differentiable in $(0, \nu)$, $\phi'_-(\nu) := \lim_{t\uparrow\nu} \phi'(t) > 0$ and $\phi(t) = 1$ for $t \in [\nu, \infty)$.

Some capped folded concave functions satisfying these two properties are discussed in Pan & Chen (2021). For the reader's ease, we present a few examples below.

(1) Capped L_1: $\phi^{\text{CapL1}}(t) = \min\left\{1, \frac{t}{\nu}\right\}$;

(2) Capped L_p: $\phi^{\text{CapLp}}(t) = \min\left\{1, \frac{t^p}{\nu^p}\right\}$, $0 < p < 1$;

(3) Capped Fraction: $\phi^{\text{CapF}}(t) = \min\left\{1, \frac{(1+\alpha\nu)t}{\nu(1+\alpha t)}\right\}$, $\alpha > 0$;

(4) Capped Logarithm: $\phi^{\text{CapLog}}(t) = \min\left\{1, \log\left(\frac{t}{\theta} + 1\right) / \log\left(\frac{\nu}{\theta} + 1\right)\right\}$, $\theta > 0$.

For simplicity, we denote

$$\begin{aligned} \varOmega &= \left\{(X, Y) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n} \mid \left\|\mathscr{A}\left(XY^T\right) - \boldsymbol{b}\right\|_2 \le \sigma\right\}, \\ F(X, Y) &= \left(\|\mathscr{A}\left(XY^T\right) - \boldsymbol{b}\|_2^2 - \sigma^2\right)_+, \quad (z)_+ = \max\{0, z\}, \ \forall z \in \mathbb{R}, \\ \varPhi(X) &= \sum_{i=1}^{s} n_i\phi\left(\|X_i\|_p\right), \ \tilde{\varPhi}(Y) = \sum_{i=1}^{s} n_i\phi\left(\|Y_i\|_p\right), \ \varPsi(X, Y) = \varPhi(X) + \tilde{\varPhi}(Y). \end{aligned}$$

Next, we present some relationships between problems (1.1), (3.1), (3.2) and (3.3).

3.1 *Link between problems (*1.1*) and (*3.1*)*

THEOREM 3. Problems (1.1) and (3.1) are equivalent. Moreover, they have the same optimal values.

Proof. First, let $\tilde{C} \in \mathbb{R}^{m\times n}$ be a global minimizer of problem (1.1) with $\tilde{r} := \mathrm{rank}(\tilde{C})$. Then there exist $(\tilde{X}, \tilde{Y}) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n}$ such that $\tilde{C} = \tilde{X}\tilde{Y}^T$ and $\|\tilde{X}\|_{p,0} = \|\tilde{Y}\|_{p,0} = \tilde{r}$. Thus,

$$\tilde{r} = \frac{1}{2}\left(\|\tilde{X}\|_{p,0} + \|\tilde{Y}\|_{p,0}\right) \geq r^*, \tag{3.4}$$

where r^* is the global minimum of problem (3.1).

Next, suppose $(X^*, Y^*) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n}$ is a global minimizer of problem (3.1), then

$$\tilde{r} \leq \mathrm{rank}\left(X^*\left(Y^*\right)^T\right) \leq \frac{1}{2}\left(\mathrm{rank}\left(X^*\right) + \mathrm{rank}\left(Y^*\right)\right) \leq \frac{1}{2}\left(\left\|X^*\right\|_{p,0} + \left\|Y^*\right\|_{p,0}\right) = r^*. \tag{3.5}$$

Hence, using (3.4) and (3.5), we ensure that problems (1.1) and (3.1) are equivalent. Moreover, they have the same optimal values. □

3.2 *Link between problems (*3.1*) and (*3.2*)*

In this subsection, we first give the nature of the feasible solution in problem (3.1).

LEMMA 1. If $(X^*, Y^*) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n}$ is a global minimizer of problem (3.1) with $\|X^*\|_{p,0} + \|Y^*\|_{p,0} = 2k$, then, for any $(X, Y) \in \Omega$, we have $\|X\|_{p,0} \geq k$ and $\|Y\|_{p,0} \geq k$. Thus, $\|X^*\|_{p,0} = \|Y^*\|_{p,0} = k$.

Proof. Assume on the contrary $\min\{\|X\|_{p,0}, \|Y\|_{p,0}\} = r < k$. Let $C = XY^T$, then we obtain $\mathrm{rank}\,(C) \leq r$ by

$$\mathrm{rank}\,(C) \leq \mathrm{rank}\,(X) \leq \|X\|_{p,0}, \ \mathrm{rank}\,(C) \leq \mathrm{rank}\,(Y) \leq \|Y\|_{p,0}.$$

Utilizing Theorem 3, we get $\mathrm{rank}\,(C) \geq k$. This contradicts $\mathrm{rank}\,(C) \leq r < k$. Hence, $\|X\|_{p,0} \geq k$ and $\|Y\|_{p,0} \geq k$. □

For integers s and t with $0 \leq s, t \leq n$, denote

$$Q_X^s := \left\{X \in \mathbb{R}^{m\times n} : \|X\|_{p,0} \leq s\right\}, \ Q_Y^t := \left\{Y \in \mathbb{R}^{n\times n} : \|Y\|_{p,0} \leq t\right\}$$

and

$$\mathrm{dist}_p\left(\Omega_X, Q_X^s\right) := \inf_{X\in\Omega_X}\left\{\mathrm{dist}_p\left(X, Q_X^s\right)\right\}, \ \mathrm{dist}_p\left(\Omega_Y, Q_Y^t\right) := \inf_{Y\in\Omega_Y}\left\{\mathrm{dist}_p\left(Y, Q_Y^t\right)\right\},$$

where

$$\Omega_X = \{X \mid \exists\, Y \text{ s.t. } (X, Y) \in \Omega\} \text{ and } \Omega_Y = \{Y \mid \exists\, X \text{ s.t. } (X, Y) \in \Omega\}.$$

Recall that the global minimum of problem (3.1) is a positive integer k. Then the feasible set Ω of problem (3.1) does not have (X, Y) with $\min\{\|X\|_{p,0}, \|Y\|_{p,0}\} < k$ by Lemma 1, which means

$\bar{\nu} = \min\left\{\bar{\nu}_X, \bar{\nu}_Y\right\} > 0$ with

$$\bar{\nu}_X = \min\left\{\frac{1}{2K}\,\mathrm{dist}_p\left(\Omega_X, Q_X^{k-K}\right) : K = 1,\ldots,k\right\},$$

$$\bar{\nu}_Y = \min\left\{\frac{1}{2K}\,\mathrm{dist}_p\left(\Omega_Y, Q_Y^{k-K}\right) : K = 1,\ldots,k\right\}.$$

In the following, we show that problems (3.1) and (3.2) have the same global optimal solutions for any $\nu \in (0, \bar{\nu})$.

THEOREM 4. For any capped folded concave function ϕ satisfying $0 < \nu < \bar{\nu}$ and $\phi^{\mathrm{CapL1}}(t) \le \phi(t) < |t|^0$ for $t \in (0, \nu)$, problems (3.1) and (3.2) have same global minimizers and same optimal values.

Proof. (*i*) Let $(X^*, Y^*) \in \mathbb{R}^{m\times n}\times\mathbb{R}^{n\times n}$ be a global minimizer of problem (3.1) with $\|X^*\|_{p,0}+\|Y^*\|_{p,0} = 2k$. We prove (X^*, Y^*) is also a global minimizer of problem (3.2) for any $0 < \nu < \bar{\nu}$. Since the global optimality of problem (3.1) yields $\|X\|_{p,0} \ge k$ and $\|Y\|_{p,0} \ge k$ for any $(X, Y) \in \Omega$ by Lemma 1, we show the conclusion by two cases:

Case 1. $\|X\|_{p,0} = k$. It is easy to see that for any $i \in \Gamma(X)$,

$$\begin{aligned}\|X_i\|_p &\ge \min\left\{\left\|X_j\right\|_p > 0 : j = 1,\ldots,s\right\}\\ &= \mathrm{dist}_p\left(X, Q_X^{k-1}\right) \ge \mathrm{dist}_p\left(\Omega_X, Q_X^{k-1}\right) \ge \bar{\nu} > \nu,\end{aligned}$$

which means that $\Phi(X) = k = \Phi(X^*)$.

Case 2. $\|X\|_{p,0} = r > k$. Without loss of generality, assume $\left\|X_1\right\|_p,\ldots,\left\|X_{i_1}\right\|_p \in (0,\nu)$, $\left\|X_{i_1+1}\right\|_p,\ldots,\left\|X_{i_2}\right\|_p \in [\nu,+\infty)$ and $\left\|X_{i_2+1}\right\|_p = \cdots = \left\|X_s\right\|_p = 0$. If $r' := n_{i_1+1} + \cdots + n_{i_2} \ge k$, from $\phi(t) > 0$ for $t > 0$, we have $\Phi(X) > k$. Now assume $r' < k$, we know that

$$\frac{1}{k-r'}\,\mathrm{dist}_p\left(\Omega_X, Q_X^{r'}\right) \ge \bar{\nu}.$$

Together with

$$\begin{aligned}n_1\left\|X_1\right\|_p + \cdots + n_{i_1}\left\|X_{i_1}\right\|_p &\ge \sqrt[p]{n_1\left\|X_1\right\|_p^p + \cdots + n_{i_1}\left\|X_{i_1}\right\|_p^p}\\ &\ge \mathrm{dist}_p\left(X, Q_X^{r'}\right) \ge \mathrm{dist}_p\left(\Omega_X, Q_X^{r'}\right),\end{aligned}$$

we get

$$\begin{aligned}\Phi(X) &= n_1\phi\left(\left\|X_1\right\|_p\right) + \cdots + n_{i_1}\phi\left(\left\|X_{i_1}\right\|_p\right) + \cdots + n_{i_2}\phi\left(\left\|X_{i_2}\right\|_p\right)\\ &\ge n_1\phi^{\mathrm{CapL1}}\left(\left\|X_1\right\|_p\right) + \cdots + n_{i_1}\phi^{\mathrm{CapL1}}\left(\left\|X_{i_1}\right\|_p\right) + r'\\ &= \frac{1}{\nu}\left(n_1\left\|X_1\right\|_p + \cdots + n_{i_1}\left\|X_{i_1}\right\|_p\right) + r'\end{aligned}$$

$$\geq \frac{1}{\nu}\,\mathrm{dist}_p\left(\Omega_X, Q_X^{r'}\right) + r'$$
$$\geq \frac{1}{\nu}\left(k - r'\right)\bar{\nu} + r'$$
$$> \frac{1}{\bar{\nu}}\left(k - r'\right)\bar{\nu} + r' = k. \tag{3.6}$$

The above two cases imply that $\Phi(X) \geq k = \Phi(X^*)$. Using similar ways in the proof for X above, we will obtain $\tilde{\Phi}(Y) \geq k = \tilde{\Phi}(Y^*)$. Hence, (X^*, Y^*) is also a global minimizer of problem (3.2). Moreover, we have $\|X^*\|_{p,0} + \|Y^*\|_{p,0} = \Phi(X^*) + \tilde{\Phi}(Y^*)$ for each global minimizer (X^*, Y^*) of problem (3.2).

(*ii*) Let $(\hat{X}, \hat{Y})$ be a global minimizer of problem (3.2) with $0 < \nu < \bar{\nu}$. Assume on the contrary $(\hat{X}, \hat{Y})$ is not a solution of problem (3.1). Let (X^*, Y^*) be a global minimizer of problem (3.1), that is, $\|X^*\|_{p,0} = \|Y^*\|_{p,0} = k$. By $\phi^{\mathrm{CapL1}}(t) \leq \phi(t) \leq |t|^0$, we have $\Phi(X^*) \leq \|X^*\|_{p,0}$ and $\tilde{\Phi}(Y^*) \leq \|Y^*\|_{p,0}$. We may assume that $\|\hat{X}\|_{p,0} > k$, since $\max\{\|\hat{X}\|_{p,0}, \|\hat{Y}\|_{p,0}\} > k$ is obvious when $(\hat{X}, \hat{Y})$ is not a solution of problem (3.1). Using similar ways in the proof for Case 2 above, we will obtain $\Phi(\hat{X}) > k = \|X^*\|_{p,0} \geq \Phi(X^*)$ for any $0 < \nu < \bar{\nu}$. Thus,

$$\Phi(\hat{X}) + \tilde{\Phi}(\hat{Y}) > 2k \geq \Phi(X^*) + \tilde{\Phi}(Y^*).$$

This contradicts the global optimality of $(\hat{X}, \hat{Y})$ for problem (3.2). Hence, $(\hat{X}, \hat{Y})$ is a global minimizer of problem (3.1).

Therefore, whenever $0 < \nu < \bar{\nu}$, problems (3.1) and (3.2) have the same global minimizers and optimal values. □

3.3 *Link between problems (*3.2*) and (*3.3*)*

THEOREM 5. Suppose that ϕ is globally Lipschitz continuous on $[0, \nu]$. Then it holds that

(1) there exists a $\mu^* > 0$ such that any global minimizer of problem (3.2) is a global minimizer of problem (3.3) whenever $\mu \geq \mu^*$;

(2) if $(\bar{X}, \bar{Y})$ is a global minimizer of problem (3.3) for some $\mu > \mu^*$, then $(\bar{X}, \bar{Y})$ is a global minimizer of problem (3.2).

Proof. Since ϕ is globally Lipschitz continuous on $[0, \nu]$, $\Psi(X, Y)$ is globally Lipschitz continuous on $[0, \nu]$. Similar to (Chen *et al.*, 2016, Lemma 3.1), we can easily obtain (1) and (2). □

3.4 *Link between problems (*3.1*) and (*3.3*)*

3.4.1 *Stationary point of problem (*3.3*).* Let $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ be locally Lipschitz continuous and directionally differentiable at point $X \in \mathbb{R}^{m\times n}$. The directional derivative of f along a matrix $W \in \mathbb{R}^{m\times n}$ at X is defined by

$$f'(X; W) := \lim_{\tau\downarrow 0} \frac{f(X + \tau W) - f(X)}{\tau}.$$

If f is differentiable at X, then $f'(X;W) = \langle \nabla f(X), W\rangle$. Denote $\bar{Z} := (\bar{X}, \bar{Y})$, by simple computation, we have

$$F'\left(\bar{Z}; Z-\bar{Z}\right) = \begin{cases} 0, & \text{if } \left\|\mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}\right\|_2^2 < \sigma^2, \\ \max\left\{0, \Delta_1 + \Delta_2\right\}, & \text{if } \left\|\mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}\right\|_2^2 = \sigma^2, \\ \Delta_1 + \Delta_2, & \text{otherwise,} \end{cases}$$

where

$$\Delta_1 = 2\left\langle \mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}, \mathscr{A}\left(\left(X-\bar{X}\right)\bar{Y}^T\right)\right\rangle,$$
$$\Delta_2 = 2\left\langle \mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}, \mathscr{A}\left(\bar{X}\left(Y-\bar{Y}\right)^T\right)\right\rangle.$$

Definition 3. We say that $\bar{Z} := \left(\bar{X}, \bar{Y}\right) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n}$ is a stationary point of problem (3.3) if

$$\Phi'\left(\bar{X}; X-\bar{X}\right) + \tilde{\Phi}'\left(\bar{Y}; Y-\bar{Y}\right) + \mu F'\left(\bar{Z}; Z-\bar{Z}\right) \geq 0, \quad \forall\,(X,Y) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n}.$$

3.4.2 *Characterizations of stationary points of problem (3.3).* We obtain from (Luo & Luo, 1994, Theorem 3.1) that there exists a $\beta > 0$ such that for all $Z = (X, Y) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n}$, we have

$$\operatorname{dist}_p(Z, \Omega) \leq \beta\left(\left\|\mathscr{A}\left(XY^T\right) - \boldsymbol{b}\right\|_2^2 - \sigma^2\right)_+ = \beta F(X,Y). \tag{3.7}$$

Let $L(X,Y) = 2\|\mathscr{A}^*\|_q \left\|\mathscr{A}\left(XY^T\right) - \boldsymbol{b}\right\|_q \max\{\|X\|_p, \|Y\|_p\}$ with

$$1/p + 1/q = 1, \quad \|\mathscr{A}^*\|_q := \sup_{\|x\|_q=1} \left\|\mathscr{A}^*(x)\right\|_q.$$

Since $\phi'_-(\nu) \to \infty$ as $\nu \to 0$, for any $\Upsilon > \sigma$ and $\mu > 0$, there are $(\hat{X}, \hat{Y})$ and a sufficiently small $\nu > 0$ such that $\|\mathscr{A}(\hat{X}\hat{Y}^T) - \boldsymbol{b}\|_q \max\{\|\hat{X}\|_p, \|\hat{Y}\|_p\} \geq \Upsilon$ and $\phi'_-(\nu) > \mu L(\hat{X}, \hat{Y})$. In the rest of this paper, we choose Υ, ν, μ and $(\hat{X}, \hat{Y})$ satisfying

$$\|\mathscr{A}(\hat{X}\hat{Y}^T) - \boldsymbol{b}\|_q \max\{\|\hat{X}\|_p, \|\hat{Y}\|_p\} \geq \Upsilon, \quad \mu > \beta/\bar{\nu}, \quad \phi'_-(\nu) > \lambda L(\hat{X}, \hat{Y}).$$

We then show a lower bound property of the stationary points of problem (3.3).

Lemma 2. Let $(\bar{X}, \bar{Y}) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{n\times n}$ be a stationary point of problem (3.3) satisfying $\left\|\mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}\right\|_q \max\{\|\bar{X}\|_p, \|\bar{Y}\|_p\} \leq \Upsilon$ and $\phi'_-(\nu) > \lambda L(\hat{X}, \hat{Y})$. Then for $i = 1, \ldots, s$, we have

$$\begin{cases} \|\bar{X}_i\|_p \geq \nu \\ \|\bar{Y}_i\|_p \geq \nu \end{cases} \quad \text{or} \quad \begin{cases} \|\bar{X}_i\|_p = 0 \\ \|\bar{Y}_i\|_p = 0. \end{cases}$$

Proof. To prove this Lemma, we only need to show $\Gamma_1(\bar{X}) \cup \Gamma_2(\bar{Y}) = \emptyset$. Assume on contradiction that $\Gamma_1(\bar{X}) \cup \Gamma_2(\bar{Y}) \neq \emptyset$. Might as well set $\Gamma_1(\bar{X}) \neq \emptyset$. From Definition 3, we have the following inequality for any (X, Y) satisfying $Y = \bar{Y}, X_j = \bar{X}_j$ for all $j \notin \Gamma_1(\bar{X})$ and for which $\exists\, i \in \Gamma_1(\bar{X})$ such that $X_i = \bar{X}_i - \frac{\varepsilon}{n_i}\bar{X}_i$ with $\varepsilon > 0$,

$$\begin{aligned}\varepsilon\phi'\left(\|\bar{X}_i\|_p\right)\|\bar{X}_i\|_p \leq& \mu F'\left(\bar{Z}; Z - \bar{Z}\right) \leq \mu\left|\Delta_1 + \Delta_2\right| \\ \leq& 2\mu\varepsilon\left\|\mathscr{A}^*\left(\mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}\right)\right\|_q \|\bar{X}_i\|_p\|\bar{Y}_i\|_p.\end{aligned}$$

Thus,

$$\begin{aligned}\phi'_-(\nu) \leq \phi'\left(\|\bar{X}_i\|_p\right) &\leq 2\mu\left\|\mathscr{A}^*\left(\mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}\right)\right\|_q \|\bar{Y}_i\|_p \\ &\leq 2\mu\left\|\mathscr{A}^*\right\|_q \Upsilon \leq \mu L(\hat{X}, \hat{Y}),\end{aligned}$$

where the first inequality follows from $i \in \Gamma_1(\bar{X})$.

This contradicts the condition of $\phi'_-(\nu) > \mu L(\hat{X}, \hat{Y})$. The proof is completed. □

THEOREM 6. Let $\mu > \beta/\bar{\nu}$ and $\phi'_-(\nu) > \mu L(\hat{X}, \hat{Y})$. Then the following statements hold:

(1) If $(\bar{X}, \bar{Y})$ is a global minimizer of problem (3.3) with $\left\|\mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}\right\|_q \max\{\|\bar{X}\|_p, \|\bar{Y}\|_p\} \leq \Upsilon$, then $(\bar{X}, \bar{Y})$ is a global minimizer of problem (3.1).

(2) If (X^*, Y^*) is a global minimizer of problem (3.1) and problem (3.3) has a global minimizer $(\bar{X}, \bar{Y})$ with $\|\mathscr{A}(\bar{X}\bar{Y}^T) - \boldsymbol{b}\|_q \max\{\|\bar{X}\|_p, \|\bar{Y}\|_p\} \leq \Upsilon$, then (X^*, Y^*) is a global minimizer of problem (3.3).

Proof. (1) Since $(\bar{X}, \bar{Y})$ is a global minimizer of problem (3.3) and the objective function is locally Lipschitz continuous, $(\bar{X}, \bar{Y})$ is a stationary point of problem (3.3). From $\left\|\mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}\right\|_q \max\{\|\bar{X}\|_p, \|\bar{Y}\|_p\} \leq \Upsilon$ and Lemma 2, $\Phi(\bar{X}) = \|\bar{X}\|_{p,0}$ and $\tilde{\Phi}(\bar{Y}) = \|\bar{Y}\|_{p,0}$. Assume now that $(\bar{X}, \bar{Y})$ is not a global minimizer of problem (3.1) and (X^*, Y^*) with $\|X^*\|_{p,0} + \|Y^*\|_{p,0} = 2k$ is a global minimizer of problem (3.1).

We split the proof into two cases:

- If $(\bar{X}, \bar{Y}) \in \Omega$, then

$$\|X^*\|_{p,0} + \|Y^*\|_{p,0} < \|\bar{X}\|_{p,0} + \|\bar{Y}\|_{p,0}.$$

Thanks to $F(X^*, Y^*) = 0$, we have

$$\begin{aligned}\Phi\left(X^*\right) + \tilde{\Phi}\left(Y^*\right) + \mu F\left(X^*, Y^*\right) &\leq \left\|X^*\right\|_{p,0} + \left\|Y^*\right\|_{p,0} + \mu F\left(X^*, Y^*\right) \\ &= \left\|X^*\right\|_{p,0} + \left\|Y^*\right\|_{p,0} < \|\bar{X}\|_{p,0} + \|\bar{Y}\|_{p,0} = \Phi(\bar{X}) + \tilde{\Phi}(\bar{Y}) + \mu F(\bar{X}, \bar{Y}),\end{aligned}$$

which contradicts the global optimality of $(\bar{X}, \bar{Y})$ for problem (3.3).

- If $(\bar{X}, \bar{Y}) \notin \Omega$, then $F(\bar{X}, \bar{Y}) > 0$. Then, we distinguish two cases:

– if $\|\bar{X}\|_{p,0} + \|\bar{Y}\|_{p,0} \geq 2k$, we obtain that

$$\begin{aligned}\Phi\left(X^*\right) + \tilde{\Phi}\left(Y^*\right) + \mu F\left(X^*, Y^*\right) &\leq \left\|X^*\right\|_{p,0} + \left\|Y^*\right\|_{p,0} + \mu F\left(X^*, Y^*\right)\\ &= 2k < \Phi(\bar{X}) + \tilde{\Phi}(\bar{Y}) + \mu F(\bar{X}, \bar{Y}),\end{aligned}$$

which contradicts the global optimality of $(\bar{X}, \bar{Y})$ for problem (3.3).

– If $\|\bar{X}\|_{p,0} + \|\bar{Y}\|_{p,0} < 2k$, then $\min\{\|\bar{X}\|_{p,0}, \|\bar{Y}\|_{p,0}\} < k$, might as well set

$$\|\bar{X}\|_{p,0} = \min\left\{\|\bar{X}\|_{p,0}, \|\bar{Y}\|_{p,0}\right\} = k' < k.$$

Thus,

$$\begin{aligned}2\bar{\nu} &\leq \frac{1}{k-k'}\operatorname{dist}_p\left(\Omega_X, \mathcal{Q}_{\bar{X}}^{k'}\right) \leq \frac{1}{k-k'}\operatorname{dist}_p\left(\Omega_X, \bar{X}\right)\\ &\leq \frac{1}{k-k'}\operatorname{dist}_p\left(\Omega, \left(\bar{X}, \bar{Y}\right)\right) \leq \frac{\beta F(\bar{X}, \bar{Y})}{k-k'},\end{aligned}$$

where the first inequality follows from the definition of $\bar{\nu}$ and the last inequality uses (3.7). This together with $\mu > \beta/\bar{\nu}$ implies that

$$\begin{aligned}\Phi\left(\bar{X}\right) + \tilde{\Phi}\left(\bar{Y}\right) + \mu F\left(\bar{X}, \bar{Y}\right) &> 2k' + \frac{\beta}{\bar{\nu}} F\left(\bar{X}, \bar{Y}\right)\\ &\geq 2k = \left\|X^*\right\|_{p,0} + \left\|Y^*\right\|_{p,0} + \mu F\left(X^*, Y^*\right)\\ &\geq \Phi\left(X^*\right) + \tilde{\Phi}\left(Y^*\right) + \mu F\left(X^*, Y^*\right).\end{aligned}$$

This contradicts the global optimality of $(\bar{X}, \bar{Y})$ for problem (3.3).

This shows that $(\bar{X}, \bar{Y})$ is a global minimizer of problem (3.1).

(2) Suppose that (X^*, Y^*) is a global minimizer of problem (3.1), but not a global minimizer of problem (3.3). Since $(\bar{X}, \bar{Y})$ is a global minimizer of problem (3.3) with $\left\|\mathscr{A}\left(\bar{X}\bar{Y}^T\right) - \boldsymbol{b}\right\|_q \max\{\|\bar{X}\|_p, \|\bar{Y}\|_p\} \leq \Upsilon$, from Lemma 2 and result (1) of this theorem, we have

$$\Phi\left(\bar{X}\right) = \|\bar{X}\|_{p,0},\ \tilde{\Phi}\left(\bar{Y}\right) = \|\bar{Y}\|_{p,0}, \quad \left(\bar{X}, \bar{Y}\right) \in \Omega.$$

Using this, we conclude that

$$\begin{aligned}\|\bar{X}\|_{p,0} + \|\bar{Y}\|_{p,0} &\leq \|\bar{X}\|_{p,0} + \|\bar{Y}\|_{p,0} + \mu F\left(\bar{X}, \bar{Y}\right)\\ &= \Phi\left(\bar{X}\right) + \tilde{\Phi}\left(\bar{Y}\right) + \mu F\left(\bar{X}, \bar{Y}\right)\\ &< \Phi\left(X^*\right) + \tilde{\Phi}\left(Y^*\right) + \mu F\left(X^*, Y^*\right)\\ &\leq \left\|X^*\right\|_{p,0} + \left\|Y^*\right\|_{p,0},\end{aligned}$$

which leads to a contradiction with the global optimality of (X^*, Y^*) for problem (3.1). Hence, (X^*, Y^*) is a global minimizer of problem (3.3) and the proof is completed. □

4. An inexact restarted augmented Lagrangian method with the extrapolated linearized alternating minimization

By introducing the auxiliary variable $C = XY^T$, problem (3.2) can be reformulated into the following problem:

$$\min_{X,Y,C} \varPhi(X) + \tilde{\varPhi}(Y) + l_{\varTheta}(C), \quad \text{s.t.} \quad C = XY^T, \tag{4.1}$$

where $l_{\varTheta}(C)$ is an indicator function with $\varTheta = \left\{C \mid \|\mathscr{A}(C) - \boldsymbol{b}\|_2 \le \sigma\right\}$.

Problem (4.1) is to minimize a nonsmooth nonconvex function with bilinear constraints. By exploring the structure of problem (4.1), we propose an inexact restarted augmented Lagrangian (IRAL) framework in Subsection 4.1. Next, we propose an extrapolated linearized alternating minimization (ELAM) algorithm to solve the augmented Lagrangian subproblem in Subsection 4.2. Combining these two components, we name our new algorithm IRAL-ELAM, and provide a complexity analysis in Subsection 4.3. In Subsections 4.4 and 4.5, we prove that every sequence generated by IRAL-ELAM has at least one accumulation point and that each accumulation point is a stationary point of problem (4.1).

4.1 *The proposed IRAL*

In this subsection, we propose an IRAL framework to solve problem (4.1). It is easy to deduce that the augmented Lagrangian function for problem (4.1) is

$$L_\eta(X, Y, C; S) = \varPhi(X) + \tilde{\varPhi}(Y) + l_{\varTheta}(C) + \left\langle XY^T - C, S\right\rangle + \frac{\eta}{2}\left\|XY^T - C\right\|_F^2, \tag{4.2}$$

where $\eta > 0$ is the penalty parameter and S is the Lagrange multiplier matrix.

Based on the classical augmented Lagrangian method, we use the following subproblem to approximate problem (4.1) at each outer iteration:

$$\min_{X,Y,C} L_{\eta^k}\left(X, Y, C; S^k\right). \tag{4.3}$$

At the $(k+1)$th iteration, we inexactly solve problem (4.3) to obtain an approximate solution $(X^{k+1}, Y^{k+1}, C^{k+1})$ satisfying the following condition:

$$\text{dist}_2\left(0, \partial L_{\eta^k}\left(X^{k+1}, Y^{k+1}, C^{k+1}; S^k\right)\right) \le \epsilon_k. \tag{4.4}$$

Now, we present the IRAL framework for solving problem (4.1) as follows.

Remark 3. Different from the existing inexact augmented Lagrangian methods for nonsmooth nonconvex optimization problems Lu & Zhang (2012); Chen *et al.* (2017); Liu *et al.* (2019, 2023), we design a new rule for updating the Lagrangian penalty parameter and the Lagrange multiplier matrix.

Algorithm 1 IRAL algorithm for problem (4.1)

Input: Initial points X^0, Y^0, C^0, S^0. Parameters $\eta^0 > 0$, $\rho_1, \rho_2, \rho_3 \in (0,1)$, $\vartheta \in \mathbb{N}_+$ and $\epsilon_0 > 0$. Let $k := 0$.

while a stopping criterion is not met **do**

Step 1. Solve problem (4.3) to obtain $\left(X^{k+1}, Y^{k+1}, C^{k+1}\right)$ satisfying (4.4).

Step 2. If $k \le \vartheta$, set $S^{k+1} = 0$, $\eta^{k+1} = \eta^k$ and $\epsilon_{k+1} = \epsilon_k$.

Else if $k > \vartheta$ and

$$\left\| X^{k+1}(Y^{k+1})^T - C^{k+1} \right\|_F \le \rho_1 \min_{t=k-\vartheta+1,\ldots,k} \left\| X^t(Y^t)^T - C^t \right\|_F, \tag{4.5}$$

then set $S^{k+1} = 0$, $\eta^{k+1} = \eta^k$ and $\epsilon_{k+1} = \sqrt{\rho_1}\epsilon_k$.

Otherwise, compute multipliers S^{k+1} by

$$S^{k+1} = S^k + \eta^k \left(X^{k+1}(Y^{k+1})^T - C^{k+1} \right), \tag{4.6}$$

and set

$$\eta^{k+1} = \eta^k/\rho_2 \ \text{ and } \ \epsilon_{k+1} = \rho_3 \epsilon_k. \tag{4.7}$$

Step 3. Let $k := k + 1$ and go to ***Step* 1**.

end while

Output: X^{k+1}, Y^{k+1}, C^{k+1}.

4.2 *The proposed ELAM*

We now shall discuss how to solve subproblem (4.3), which is a nonconvex problem. For convenience of notation, we use $\left(X^{(k+1,\jmath)}, Y^{(k+1,\jmath)}, C^{(k+1,\jmath)}\right)$ to denote the $\jmath$th iterate of the ELAM algorithm and the $(k+1)$th iterate of Algorithm 1. For the superscript k (and $k+1$), we further denote $\eta^k, S^k, X^{(k+1,\jmath)}, Y^{(k+1,\jmath)}, C^{(k+1,\jmath)}$ by $\eta, S, X^{(\jmath)}, Y^{(\jmath)}, C^{(\jmath)}$. We assume to be at the $\jmath$th iterate of the ELAM algorithm.

(1) Computing $X_i^{\jmath+1}$: fixing other variables except for X_i in (4.2), we update $X_i^{\jmath+1}$ by the following sub-problem:

$$X_i^{\jmath+1} \leftarrow \arg\min_{X_i} \ n_i \phi\left(\|X_i\|_p \right) + \frac{\eta}{2} \left\| X_i (Y_i^{\jmath})^T - G_i^{\jmath} \right\|_F^2, \tag{4.8}$$

where

$$G_i^{\jmath} = C^{\jmath} - \sum_{l=1}^{i-1} X_l^{\jmath+1}(Y_l^{\jmath+1})^T - \sum_{l=i+1}^{s} X_l^{\jmath}(Y_l^{\jmath})^T - \frac{S}{\eta}. \tag{4.9}$$

Next, we propose a new acceleration method to solve problem (4.8). First, we give an extrapolated point $\bar{X}_i^J = X_i^J + w_{X_i}^J(X_i^J - X_i^{J-1})$. Then we solve X_i^{J+1} by solving the following problem:

$$\begin{aligned} X_i^{J+1} &= \arg\min_{X_i} \; n_i\phi\left(\left\|X_i\right\|_p\right) + \frac{\sigma_{X_i}^J}{2}\left\|X_i - \left(\bar{X}_i^J - \left(\bar{X}_i^J(Y_i^J)^T - G_i^J\right)Y_i^J/\tau_{X_i}^J\right)\right\|_F^2 \\ &= \operatorname{prox}_{n_i/\sigma_{X_i}^J}^{\phi(\|\cdot\|_p)}\left(\bar{X}_i^J - \left(\bar{X}_i^J(Y_i^J)^T - G_i^J\right)Y_i^J/\tau_{X_i}^J\right), \end{aligned} \tag{4.10}$$

where $\sigma_{X_i}^J = \eta\tau_{X_i}^J$. Based on the following lemma, there is a closed-form solution for (4.10) when $p = 2$.

LEMMA 3. (Zhang *et al.*, 2022, Lemma 1) For a positive numbers λ, the proximal operator of $\operatorname{prox}_\lambda^{\phi(\|\cdot\|_F)}(Z)$ has a closed-form solution, i.e.,

$$X^\star = \operatorname{prox}_\lambda^{\phi(\|\cdot\|_F)}(Z) := \arg\min_X \left\{\lambda\phi\left(\|X\|_F\right) + \frac{1}{2}\|X - Z\|_F^2\right\} = \begin{cases} \psi\left(\|Z\|_F\right)\frac{Z}{\|Z\|_F}, & Z \neq 0, \\ 0, \quad Z = 0, \end{cases}$$

where

$$\psi(z) \in \operatorname{prox}_{\lambda\phi}(z) := \arg\min_{x\in\mathbb{R}_+}\left\{\lambda\phi(x) + \frac{1}{2}(x - z)^2\right\}.$$

(2) Computing Y_i^{J+1}: fixing other variables except for Y_i in (4.2), we update Y_i^{J+1} by the following sub-problem:

$$Y_i^{J+1} \leftarrow \arg\min_{Y_i} \; n_i\phi\left(\left\|Y_i\right\|_p\right) + \frac{\eta}{2}\left\|X_i^{J+1}Y_i^T - G_i^J\right\|_F^2. \tag{4.11}$$

Likewise, we can obtain Y_i^{J+1} through the linearization and the proximal algorithm:

$$\begin{aligned} Y_i^{J+1} &= \arg\min_{Y_i} \; n_i\phi\left(\left\|Y_i\right\|_p\right) + \frac{\sigma_{Y_i}^J}{2}\left\|Y_i - \left(\bar{Y}_i^J - \left(X_i^{J+1}(\bar{Y}_i^J)^T - G_i^J\right)^T X_i^{J+1}/\tau_{Y_i}^J\right)\right\|_F^2 \\ &= \operatorname{prox}_{n_i/\sigma_{Y_i}^J}^{\phi(\|\cdot\|_p)}\left(\bar{Y}_i^J - \left(X_i^{J+1}(\bar{Y}_i^J)^T - G_i^J\right)^T X_i^{J+1}/\tau_{Y_i}^J\right), \end{aligned} \tag{4.12}$$

where $\sigma_{Y_i}^J = \eta\tau_{Y_i}^J$.

(3) Computing C^{J+1}: fixing other variables except for C in (4.2), we update C^{J+1} by the following sub-problem:

$$C^{J+1} \leftarrow \arg\min_C \; \frac{\eta}{2}\left\|C - X^{J+1}(Y^{J+1})^T - \frac{S}{\eta}\right\|_F^2, \quad \text{s.t.} \quad \|\mathscr{A}(C) - \boldsymbol{b}\|_2 \leq \sigma. \tag{4.13}$$

Thus, the optimal solution of (4.13) is

$$C^{J+1} = \Pi_{\Theta}\left(X^{J+1}(Y^{J+1})^T + \frac{S}{\eta}\right), \tag{4.14}$$

where Π_{Θ} denotes the projection onto set Θ.

We summary the solving algorithm for subproblem (4.3) in Algorithm 2.

Algorithm 2 ELAM algorithm for subproblem (4.3)

Input: Initial points X^0, Y^0. Let $X^{-1} := X^0$, $Y^{-1} := Y^0$. Let $J := 0$.

for $i = 1, \ldots, s$ **do**

Step 1. Compute $w_{x_i}^J$ and $\bar{X}_i^J = X_i^J + w_{x_i}^J\left(X_i^J - X_i^{J-1}\right)$.

Step 2. Compute X_i^{J+1} by (4.10).

Step 3. Compute $w_{y_i}^J$ and $\bar{Y}_i^J = Y_i^J + w_{y_i}^J\left(Y_i^J - Y_i^{J-1}\right)$.

Step 4. Compute Y_i^{J+1} by (4.12).

end for

Step 5. Remove the zero columns of X^{J+1} and Y^{J+1}.

Step 6. Compute C^{J+1} by (4.14).

Step 7. Let $J := J + 1$. If the stop criterion is not met, return to ***Step*** **1**.

Output: X^J, Y^J, C^J.

Remark 4. In this paper, we set $\tau_{X_i}^0 = \tau_{Y_i}^0 = 1$, and for any $J \geq 1$, let $\tau_{X_i}^J = \max\{\gamma\|Y_i^J\|^2, \varepsilon\}$ and $\tau_{Y_i}^J = \max\{\gamma\|X_i^{J+1}\|^2, \varepsilon\}$ with $\gamma > 1$. In addition, we take

$$w_{x_i}^J = \min\left\{\hat{\gamma}_i^J, \frac{\delta(\gamma-1)}{2(\gamma+1)}\sqrt{\frac{\tau_{X_i}^{J-1}}{\tau_{X_i}^J}}\right\}, \quad w_{y_i}^J = \min\left\{\hat{\gamma}_i^J, \frac{\delta(\gamma-1)}{2(\gamma+1)}\sqrt{\frac{\tau_{Y_i}^{J-1}}{\tau_{Y_i}^J}}\right\},$$

where $\delta < 1$ and

$$\hat{\gamma}_i^J = \frac{t_{i-1}^J - 1}{t_i^J}, \quad t_0^1 = 1, t_0^J = t_s^{J-1}, \text{ for } J \geq 2,$$

$$t_i^J = \frac{1}{2}\left(1 + \sqrt{1 + 4\left(t_{i-1}^J\right)^2}\right), \text{ for } J \geq 1, i = 1, \ldots, s.$$

4.3 *Complexity analysis*

The computational cost of spectral function regularization algorithms, such as SVT Cai *et al.* (2010), is dominated by SVD, with a complexity of $\mathcal{O}(mn\min\{m,n\})$. Based on matrix factorization, NMFC Xu *et al.* (2012) has the cost $\mathcal{O}(mnr_0)$, where r_0 is the preestimated rank. In contrast, group sparsity-based methods, such as FGSR Fan *et al.* (2019), GUIG Jia *et al.* (2021) and our proposed FLGSR, avoid the time-consuming computation of SVD by focusing on matrix multiplication. This approach allows for adaptive adjustment of the matrix rank, achieving a complexity of $\mathcal{O}(mnr_J)$ when the iterates X^J and Y^J have r_J nonzero columns.

Although the number of groups in our proposed algorithm does not affect its theoretical complexity, the computational time increases with the number of groups. To mitigate excessive computational time, current group sparsity-based algorithms (e.g., FGSR, GUIG) typically iterate over the entire variables X and Y instead of treating each group X_i and Y_i as separate variables. However, this approach fails to utilize the updated values X_i^l and Y_i^l $(l \leq J)$ when computing X_i^{J+1} and Y_i^{J+1}, missing out on potential improvements in convergence and accuracy. This is similar to the difference between the Jacobi and Gauss-Seidel algorithms for solving linear systems, where the latter benefits from using the latest updates to enhance convergence and accuracy. Our flexible grouping method addresses this issue by maintaining a small number of groups, regardless of matrix column size, allowing X_i and Y_i to be treated as separate variables without significantly increasing computational time. For a detailed analysis of how the number of groups affects computational efficiency and result quality, refer to Subsection 5.1.1.

4.4 *Convergence analysis of Algorithm* 1

The following theorem states the main result of our convergence analysis for the proposed Algorithm 1.

Theorem 7. Suppose that the sequence $\{X^k, Y^k, C^k, S^k\}_{k\in\mathbb{N}^+}$ generated by Algorithm 1 is bounded. Then the following statements hold:

(i) $\lim_{k\to\infty} \left\|X^k(Y^k)^T - C^k\right\|_F = 0$;

(ii) any accumulation point $(X^\star, Y^\star, C^\star, S^\star)$ of $\{X^k, Y^k, C^k, S^k\}_{k\in\mathbb{N}^+}$ is a stationary point of problem (4.1).

Proof. (i) We split the proof into two cases:

Case 1. The sequence $\{\eta^k\}$ is bounded. In this case, (4.7) happens finite times at most, which means that there exists $k_0 > \vartheta$ such that $\eta^k = \eta^{k_0}$ and

$$\left\|X^{k+1}(Y^{k+1})^T - C^{k+1}\right\|_F \leq \rho_1 \min_{t=k-\vartheta+1,\ldots,k} \left\|X^t(Y^t)^T - C^t\right\|_F$$

for all $k > k_0$. We obtain from the above inequality that for all $k > k_0$,

$$\left\|X^{k+1}(Y^{k+1})^T - C^{k+1}\right\|_F \leq \rho_1 \left\|X^k(Y^k)^T - C^k\right\|_F.$$

Together this inequality with $\rho_1 \in (0,1)$ implies that $\lim_{k\to\infty} \|X^k(Y^k)^T - C^k\|_F = 0$.

Case 2. The sequence $\{\eta^k\}$ is unbounded. In this case, the set

$$\mathcal{K} = \left\{k : \eta^{k+1} = \eta^k/\rho_2\right\} \tag{4.15}$$

is infinite. Thanks to $\rho_2 \in (0,1)$, (4.15) leads to $\lim_{k\to\infty, k\in\mathcal{K}} \eta^k = \infty$. Given $k > \vartheta$, let t_k be the largest element in $\mathcal{K}$ satisfying $t_k \leq k$. Subsequently, we demonstrate that

$$\left\|X^{k+1}(Y^{k+1})^T - C^{k+1}\right\|_F \leq \frac{\|S^{t_k}\|_F}{\eta^{t_k}} + \frac{\|S^{t_k+1}\|_F}{\eta^{t_k}}. \tag{4.16}$$

It is clear that the inequality (4.16) holds when $t_k = k$. Therefore, we only need to consider $t_k < k$ in the following. Combining (4.5) and (4.6), one has

$$\left\| X^{k+1}(Y^{k+1})^T - C^{k+1} \right\|_F \le \left\| X^k(Y^k)^T - C^k \right\|_F \\ \le \dots \le \left\| X^{t_k+1}(Y^{t_k+1})^T - C^{t_k+1} \right\|_F \le \frac{\|S^{t_k}\|_F}{\eta^{t_k}} + \frac{\|S^{t_k+1}\|_F}{\eta^{t_k}}.$$

Together with the boundedness of S^k, $t_k \in \mathcal{K}$ and $\lim_{k\to\infty, k\in\mathcal{K}} \eta^k = \infty$, we know that

$$\lim_{k\to\infty} \left\| X^{k+1}(Y^{k+1})^T - C^{k+1} \right\|_F \le \lim_{k\to\infty} \frac{\|S^{t_k}\|_F}{\eta^{t_k}} + \lim_{k\to\infty} \frac{\|S^{t_k+1}\|_F}{\eta^{t_k}} = 0.$$

The above inequality yield that $\lim_{k\to\infty} \left\| X^k(Y^k)^T - C^k \right\|_F = 0$. Thus, we complete the proof of statement (i).

(ii) From Bolzano–Weierstrass Theorem Browder (2012), $\left\{X^k, Y^k, C^k, S^k\right\}_{k\in\mathbb{N}^+}$ has at least one accumulation point $(X^\star, Y^\star, C^\star, S^\star)$ and there exists a subsequence that converges to this accumulation point. Without loss of generality, we assume that the sequence is $\left\{X^k, Y^k, C^k, S^k\right\}$. Recalling result (i) of this theorem, we obtain that this accumulation point is feasible.

Together with the inequality (4.4) and the definition of $L_\eta(X, Y, C; S)$, there exists $\zeta^{k+1} \in \partial_Z L_\eta(X, Y, C; S)$ satisfying $\|\zeta^{k+1}\|_F \le \epsilon_k$ such that

$$\zeta^{k+1} \in \partial_Z \left[\Phi\left(X^{k+1}\right) + \tilde{\Phi}\left(Y^{k+1}\right) + l_\Theta\left(C^{k+1}\right)\right] + \nabla_Z \left[\left\langle h\left(Z^{k+1}\right), S^k \right\rangle + \frac{\eta^k}{2} \left\| h\left(Z^{k+1}\right) \right\|_F^2\right] \\ = \partial_Z \left[\Phi\left(X^{k+1}\right) + \tilde{\Phi}\left(Y^{k+1}\right) + l_\Theta\left(C^{k+1}\right)\right] + \left[\bar{S}^k Y^{k+1}, (\bar{S}^k)^T X^{k+1}, -\bar{S}^k\right], \tag{4.17}$$

where $Z = (X, Y, C)$, $h(Z) = XY^T - C$ and $\bar{S}^k = S^k + \eta^k h\left(Z^{k+1}\right)$.

In view of (4.6) hold when $k \in \mathcal{K}$, we then obtain that

$$\lim_{k\in\mathcal{K}, k\to\infty} S^k + \eta^k h\left(Z^{k+1}\right) = \lim_{k\in\mathcal{K}, k\to\infty} S^{k+1}. \tag{4.18}$$

If $k \notin \mathcal{K}$, by $S^{k+1} = 0$, $\eta^{k+1} = \eta^k$ and result (i) of this theorem, we get

$$\lim_{k\notin\mathcal{K}, k\to\infty} S^k + \eta^k h\left(Z^{k+1}\right) = \lim_{k\notin\mathcal{K}, k\to\infty} S^{k+1}. \tag{4.19}$$

Combining (4.18) with (4.19), we obtain that

$$\lim_{k\to\infty} S^k + \eta^k h\left(Z^{k+1}\right) = \lim_{k\to\infty} S^{k+1}. \tag{4.20}$$

By the update rule of Algorithm 1, we have $\epsilon_k \le \max\left\{\sqrt{\rho_1}, \rho_3\right\}\epsilon_{k-1}$ for all $k > \vartheta$. Combining this with the fact that $\rho_1, \rho_3 \in (0,1)$, we obtain $\lim_{k\to\infty}\epsilon_k = 0$. Consequently, we can infer that $\lim_{k\to\infty}\zeta^{k+1} = 0$, which, together with (4.17) and (4.20), implies that

$$0 \in \partial_Z\left[\Phi\left(X^\star\right) + \tilde{\Phi}\left(Y^\star\right) + l_\Theta\left(C^\star\right)\right] + \nabla_Z\left[\left\langle X^\star(Y^\star)^T - C^\star, S^\star\right\rangle\right].$$

Hence, $(X^\star, Y^\star, C^\star, S^\star)$ is a stationary point of problem (4.1), which completes the statement (*ii*). □

4.5 *Convergence analysis of Algorithm* 2

In this subsection, we prove the convergence of Algorithm 2 for solving subproblem (4.3). The main results are given in Theorem 8 below. We first give some lemmas.

To simply the notation, we denote $f_{X_i} := \frac{1}{2}\|X_i(Y_i^J)^T - G_i^J\|_F^2$ and $f_{Y_i} := \frac{1}{2}\|X_i^{J+1}Y_i^T - G_i^J\|_F^2$ in this subsection.

LEMMA 4. Let $\{X^J, Y^J, C^J\}$ be the sequence generated by Algorithm 2, then

$$\begin{aligned}
&L_\eta\left(X^{J+1}, Y^{J+1}, C^{J+1}; S\right) - L_\eta\left(X^J, Y^J, C^J; S\right)\\
&\le -\frac{\eta}{2}\|C^{J+1} - C^J\|_F^2 + \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{X_i}^{J-1}\delta^2\|X_i^J - X_i^{J-1}\|_F^2 - \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{X_i}^{J}\|X_i^{J+1} - X_i^J\|_F^2\\
&\quad + \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{Y_i}^{J-1}\delta^2\|Y_i^J - Y_i^{J-1}\|_F^2 - \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{Y_i}^{J}\|Y_i^{J+1} - Y_i^J\|_F^2.
\end{aligned}$$

Proof. From the Lipschitz continuity of $\nabla_{X_i}f_{X_i}$ about X_i, it holds that

$$f_{X_i^{J+1}} \le f_{X_i^J} + \left\langle \nabla_{X_i}f_{X_i^J}, X_i^{J+1} - X_i^J\right\rangle + \frac{\tau_{X_i}^J}{2\gamma}\|X_i^{J+1} - X_i^J\|_F^2. \tag{4.21}$$

Since X_i^{J+1} is the minimizer of (4.10), then

$$\begin{aligned}
&n_i\phi\left(\|X_i^{J+1}\|_p\right) + \frac{\sigma_{X_i}^J}{2}\|X_i^{J+1} - \bar{X}_i^J\|_F^2 + \eta\left\langle X_i^{J+1} - \bar{X}_i^J, \nabla_{X_i}f_{\bar{X}_i^J}\right\rangle\\
&\qquad \le n_i\phi\left(\|X_i^J\|_p\right) + \frac{\sigma_{X_i}^J}{2}\|X_i^J - \bar{X}_i^J\|_F^2 + \eta\left\langle X_i^J - \bar{X}_i^J, \nabla_{X_i}f_{\bar{X}_i^J}\right\rangle.
\end{aligned} \tag{4.22}$$

By summing (4.21) and (4.22), and letting $\Delta X_i^J := X_i^{J+1} - X_i^J$, we have

$$\begin{aligned}
&n_i\phi\left(\|X_i^{J+1}\|_p\right) + \eta f_{X_i^{J+1}} - n_i\phi\left(\|X_i^J\|_p\right) - \eta f_{X_i^J} \\
&\quad\le \eta\left\langle \nabla_{X_i} f_{X_i^J} - \nabla_{X_i} f_{\bar{X}_i^J}, \Delta X_i^J\right\rangle + \frac{\sigma_{X_i}^J}{2\gamma}\|\Delta X_i^J\|_F^2 + \frac{\sigma_{X_i}^J}{2}\|X_i^J - \bar{X}_i^J\|_F^2 - \frac{\sigma_{X_i}^J}{2}\|X_i^{J+1} - \bar{X}_i^J\|_F^2 \\
&\quad= \eta\left\langle \nabla_{X_i} f_{X_i^J} - \nabla_{X_i} f_{\bar{X}_i^J}, \Delta X_i^J\right\rangle + \sigma_{X_i}^J\left\langle -\Delta X_i^J, X_i^J - \bar{X}_i^J\right\rangle - \frac{\sigma_{X_i}^J(\gamma-1)}{2\gamma}\left\|\Delta X_i^J\right\|_F^2 \\
&\quad\le \eta\left\|\Delta X_i^J\right\|_F\left(\left\|\nabla_{X_i} f_{X_i^J} - \nabla_{X_i} f_{\bar{X}_i^J}\right\|_F + \tau_{X_i}^J\left\|X_i^J - \bar{X}_i^J\right\|_F\right) - \frac{\sigma_{X_i}^J(\gamma-1)}{2\gamma}\left\|\Delta X_i^J\right\|_F^2 \\
&\quad\le \frac{\sigma_{X_i}^J(1+\gamma)}{\gamma}\left\|\Delta X_i^J\right\|_F\left\|X_i^J - \bar{X}_i^J\right\|_F - \frac{\sigma_{X_i}^J(\gamma-1)}{2\gamma}\left\|\Delta X_i^J\right\|_F^2 \\
&\quad= \frac{\sigma_{X_i}^J w_{x_i}^J(1+\gamma)}{\gamma}\left\|\Delta X_i^J\right\|_F\left\|\Delta X_i^{J-1}\right\|_F - \frac{\sigma_{X_i}^J(\gamma-1)}{2\gamma}\left\|\Delta X_i^J\right\|_F^2 \\
&\quad\le \frac{\eta\tau_{X_i}^J(1+\gamma)^2}{\gamma(\gamma-1)}\left(w_{x_i}^J\right)^2\left\|\Delta X_i^{J-1}\right\|_F^2 - \frac{\eta\tau_{X_i}^J(\gamma-1)}{4\gamma}\left\|\Delta X_i^J\right\|_F^2 \\
&\quad\le \frac{\eta\tau_{X_i}^{J-1}(\gamma-1)}{4\gamma}\delta^2\left\|X_i^J - X_i^{J-1}\right\|_F^2 - \frac{\eta\tau_{X_i}^J(\gamma-1)}{4\gamma}\left\|X_i^{J+1} - X_i^J\right\|_F^2.
\end{aligned} \tag{4.23}$$

Here, we have used Cauchy–Schwarz inequality in the second inequality, Lipschitz continuity of $\nabla_{X_i} f_{X_i}$ about X_i in the third one, the Young's inequality in the fourth one and $w_{x_i}^J \le \frac{\delta(\gamma-1)}{2(\gamma+1)}\sqrt{\frac{\tau_{X_i}^{J-1}}{\tau_{X_i}^J}}$ to get the last inequality, the fact $\bar{X}_i^J = X_i^J + w_{x_i}^J\left(X_i^J - X_i^{J-1}\right)$ to have the second equality. Similarly, for Y_i^{J+1}, we have

$$\begin{aligned}
&n_i\phi\left(\|Y_i^{J+1}\|_p\right) + \eta f_{Y_i^{J+1}} - n_i\phi\left(\left\|Y_i^J\right\|_p\right) - \eta f_{Y_i^J} \\
&\qquad\qquad \le \frac{\eta\tau_{Y_i}^{J-1}(\gamma-1)}{4\gamma}\delta^2\left\|Y_i^J - Y_i^{J-1}\right\|_F^2 - \frac{\eta\tau_{Y_i}^J(\gamma-1)}{4\gamma}\left\|Y_i^{J+1} - Y_i^J\right\|_F^2.
\end{aligned} \tag{4.24}$$

Summing up (4.23) and (4.24) over i from 1 to s, and letting $\Delta\Psi\left(X^J, Y^J\right) := \Psi\left(X^{J+1}, Y^{J+1}\right) - \Psi\left(X^J, Y^J\right)$, one has

$$\begin{aligned}
&L_\eta\left(X^{J+1}, Y^{J+1}, C^J; S\right) - L_\eta\left(X^J, Y^J, C^J; S\right) \\
&\qquad = \Delta\Psi\left(X^J, Y^J\right) + \frac{\eta}{2}\left\|X^{J+1}\left(Y^{J+1}\right)^T - C^J + \frac{S}{\eta}\right\|_F^2 - \frac{\eta}{2}\left\|X^J\left(Y^J\right)^T - C^J + \frac{S}{\eta}\right\|_F^2 \\
&\qquad = \Delta\Psi\left(X^J, Y^J\right) + \eta\left(f_{Y_s^{J+1}} - f_{X_1^J}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \Delta\Psi\left(X^J, Y^J\right) + \eta\left(f_{Y_s^{J+1}} - f_{X_1^J}\right) + \eta\sum_{i=1}^{s-1}\left(f_{Y_i^{J+1}} - f_{X_{i+1}^J}\right) + \eta\sum_{i=1}^{s}\left(f_{X_i^{J+1}} - f_{Y_i^J}\right)\\
&= \sum_{i=1}^{s}\left[n_i\phi\left(\|X_i^{J+1}\|_p\right) + \eta f_{X_i^{J+1}} - n_i\phi\left(\|X_i^J\|_p\right) - \eta f_{X_i^J}\right]\\
&\quad + \sum_{i=1}^{s}\left[n_i\phi\left(\|Y_i^{J+1}\|_p\right) + \eta f_{Y_i^{J+1}} - n_i\phi\left(\|Y_i^J\|_p\right) - \eta f_{Y_i^J}\right]\\
&\le \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{X_i}^{J-1}\delta^2\|X_i^J - X_i^{J-1}\|_F^2 - \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{X_i}^{J}\|X_i^{J+1} - X_i^J\|_F^2\\
&\quad + \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{Y_i}^{J-1}\delta^2\|Y_i^J - Y_i^{J-1}\|_F^2 - \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{Y_i}^{J}\|Y_i^{J+1} - Y_i^J\|_F^2,
\end{aligned}
\tag{4.25}
$$

where the third equality comes from $f_{Y_i^{J+1}} = f_{X_{i+1}^J}$ for $i \in [s-1]$ and $f_{X_i^{J+1}} = f_{Y_i^J}$ for $i \in [s]$.

Since C^{J+1} is optimal to (4.13), we have

$$
L_\eta\left(X^{J+1}, Y^{J+1}, C^{J+1}; S\right) - L_\eta\left(X^{J+1}, Y^{J+1}, C^J; S\right) \le -\frac{\eta}{2}\left\|C^{J+1} - C^J\right\|_F^2. \tag{4.26}
$$

Summing up (4.25) and (4.26) completes the proof. □

LEMMA 5. Let $\{X^J, Y^J, C^J\}$ be the sequence generated by Algorithm 2, then

$$
\lim_{J\to\infty}\left(X^{J+1} - X^J\right) = 0, \quad \lim_{J\to\infty}\left(Y^{J+1} - Y^J\right) = 0, \quad \lim_{J\to\infty}\left(C^{J+1} - C^J\right) = 0. \tag{4.27}
$$

Proof. By Lemma 4, one has

$$
\begin{aligned}
&L_\eta\left(X^{\mathcal{J}+1}, Y^{\mathcal{J}+1}, C^{\mathcal{J}+1}; S\right) - L_\eta\left(X^0, Y^0, C^0; S\right) + \frac{\eta}{2}\sum_{J=1}^{\mathcal{J}}\|C^{J+1} - C^J\|_F^2\\
&\le \frac{\gamma-1}{4\gamma}\sum_{J=1}^{\mathcal{J}}\sum_{i=1}^{s}\eta\tau_{X_i}^{J-1}\left(\delta^2 - 1\right)\|X_i^J - X_i^{J-1}\|_F^2 + \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{X_i}^{0}\|X_i^1 - X_i^0\|_F^2\\
&\quad + \frac{\gamma-1}{4\gamma}\sum_{J=1}^{\mathcal{J}}\sum_{i=1}^{s}\eta\tau_{Y_i}^{J-1}\left(\delta^2 - 1\right)\|Y_i^J - Y_i^{J-1}\|_F^2 + \frac{\gamma-1}{4\gamma}\sum_{i=1}^{s}\eta\tau_{Y_i}^{0}\|Y_i^1 - Y_i^0\|_F^2.
\end{aligned}
\tag{4.28}
$$

Note that

$$
\begin{aligned}
&L_\eta\left(X^{\mathcal{J}+1}, Y^{\mathcal{J}+1}, C^{\mathcal{J}+1}; S\right) \\
&\quad = \Psi\left(X^{\mathcal{J}+1}, Y^{\mathcal{J}+1}\right) + l_\Theta\left(C^{\mathcal{J}+1}\right) + \frac{\eta}{2}\left\|X^{\mathcal{J}+1}\left(Y^{\mathcal{J}+1}\right)^T - C^{\mathcal{J}+1} + \frac{S}{\eta}\right\|_F^2 - \frac{\|S\|_F^2}{2\eta} \\
&\quad \geq -\frac{\|S\|_F^2}{2\eta} > -\infty,
\end{aligned}
\tag{4.29}
$$

where the last inequality uses the boundedness of S. Combining this with (4.28) and $\tau_{X_i}^{J-1} \geq \varepsilon$, $\tau_{Y_i}^{J-1} \geq \varepsilon$ gives

$$
\sum_{J=1}^{\infty}\left\|X^{J+1} - X^J\right\|_F^2 < \infty, \quad \sum_{J=1}^{\infty}\left\|Y^{J+1} - Y^J\right\|_F^2 < \infty, \quad \sum_{J=1}^{\infty}\left\|C^{J+1} - C^J\right\|_F^2 < \infty.
$$

Therefore, $\lim_{J\to\infty}\left(X^{J+1} - X^J\right) = 0$, $\lim_{J\to\infty}\left(Y^{J+1} - Y^J\right) = 0$, $\lim_{J\to\infty}\left(C^{J+1} - C^J\right) = 0$. □

THEOREM 8. Suppose that the sequence $\{X^J, Y^J, C^J\}$ generated by Algorithm 2 is bounded. Then the sequence $\{X^J, Y^J, C^J\}$ has at least one accumulation point, and any accumulation point $\{X^\star, Y^\star, C^\star\}$ is a stationary point of the optimization problem (4.3).

Proof. From Bolzano–Weierstrass Theorem Browder (2012), Algorithm 2 has at least one accumulation point $\{X^\star, Y^\star, C^\star\}$ and there exists one sequences that converges to this accumulation point. Without loss of generality, we assume that the sequence is $\{X^J, Y^J, C^J\}$.

For the X_i-subproblem, by first-order necessary optimality condition, we get

$$
0 \in n_i\partial\phi\left(\|X_i^{J+1}\|_p\right) + \sigma_{X_i}^J\left(X_i^{J+1} - \bar{X}_i^J\right) + \eta\left(\bar{X}_i^J(Y_i^J)^T - G_i^J\right)Y_i^J. \tag{4.30}
$$

According to $\lim_{J\to\infty}\left(X^{J+1} - X^J\right) = 0$ and $\lim_{J\to\infty}\left(Y^{J+1} - Y^J\right) = 0$, we obtain that

$$
0 \in n_i\partial\phi\left(\left\|X_i^\star\right\|_p\right) + \left(\eta X^\star Y^{\star T} - \eta C^\star + S\right)Y_i^\star. \tag{4.31}
$$

For the Y_i-subproblem, by first-order necessary optimality condition, we get

$$
0 \in n_i\partial\phi\left(\|Y_i^{J+1}\|_p\right) + \sigma_{Y_i}^J\left(Y_i^{J+1} - \bar{Y}_i^J\right) + \eta\left(X_i^{J+1}\left(\bar{Y}_i^J\right)^T - G_i^J\right)^T X_i^{J+1}. \tag{4.32}
$$

According to $\lim_{J\to\infty}\left(X^{J+1} - X^J\right) = 0$ and $\lim_{J\to\infty}\left(Y^{J+1} - Y^J\right) = 0$, we obtain that

$$
0 \in n_i\partial\phi\left(\left\|Y_i^\star\right\|_p\right) + \left(\eta X^\star Y^{\star T} - \eta C^\star + S\right)^T X_i^\star. \tag{4.33}
$$

For the C-subproblem, by first-order necessary optimality condition, we get

$$0 \in \eta \left(C^{J+1} - X^{J+1}(Y^{J+1})^T - \frac{S}{\eta} \right) + N_{\Theta}\left(C^{J+1} \right).$$

Thus,

$$0 \in \eta \left(C^{\star} - X^{\star}(Y^{\star})^T - \frac{S}{\eta} \right) + N_{\Theta}\left(C^{\star} \right). \tag{4.34}$$

By (4.31), (4.33) and (4.34), we know that $\{X^{\star}, Y^{\star}, C^{\star}\}$ is a stationary point of the optimization problem (4.3), which completes the proof. □

5. Experimental results

In this section, we compare our performance to state-of-the-art matrix completion methods, including group sparsity-based methods FGSR Fan *et al.* (2019) and GUIG Jia *et al.* (2021), matrix factorization-based method NMFC Xu *et al.* (2012), nuclear norm-based method SVT Cai *et al.* (2010), scaled gradient descent-based method ScaledGD Tong *et al.* (2021) and Riemannian gradient descent-based method LRGeomCG Vandereycken (2013). In order to demonstrate the efficiency of the proposed method, we will present the results of matrix completion experiments on two typical types of matrix data, namely, grayscale image and high altitude aerial image. For our numerical experiments, we randomly select ρ entries from the index set $\{(i,j)|1 \le i \le m, 1 \le j \le n\}$ to construct the subset Ξ, subsequently defining $\mathscr{A}$ as $\mathcal{P}_{\Xi}$. The operator $\mathcal{P}_{\Xi}$ extracts matrix entries at Ξ positions and arranges them into a vector in $\mathbb{R}^p$ to match $\boldsymbol{b}$. The inverse transformation $\mathscr{A}^{-1} = \mathcal{P}_{\Xi}^{-1}$ maps a vector in $\mathbb{R}^p$ back to an $m \times n$ matrix by placing the vector's entries at the positions specified by Ξ, with all other entries set to zero. We define the sampling ratio (SR) as $\mathrm{SR} = \rho/(mn)$. As part of our quantitative evaluation, we use three numerical metrics, namely peak signal-to-noise ratio (PSNR) Wang *et al.* (2004), structural similarity index measure (SSIM) Wang *et al.* (2004) and the recovery computation time. All experiments are conducted in Matlab R2020b under Windows 11 on a desktop of a 2.50-GHz CPU and 16-GB memory. Our Matlab codes can be downloaded from the website https://github.com/quanyumath/FLGSR.

Parameter settings and initialization. The parameters of compared algorithms are set as described in their papers, and we take the best results as the final results. In FLGSR, if not specified, η^0, ϑ, ϵ_0, $[\rho_1,\ \rho_2,\ \rho_3]$ are set as 1e-3, 10, 10 and $[0.999, 0.5, 0.5]$, respectively. The capped concave function ϕ is set as $\phi^{\mathrm{CapLog}}(t)$. All matrix data are prescaled to $[0, 1]$. In our proposed algorithm, the initial points are set as follows: $X^0 := \mathrm{eye}(m,n)$, $Y^0 := \mathrm{ones}(n,n)$, $C^0 := \mathscr{A}^{-1}(\boldsymbol{b})$, $S^0 := \mathrm{zeros}(m,n)$.

5.1 *Model analysis*

In this part, we analyse the effectiveness of flexible grouping and the restarted technique in the proposed algorithm, and test it on four images from the USC-SIPI Image Database:[1] 'Peppers', 'Sailboat', 'Bridge' and 'Mandrill'. All images have a size of 512×512. The SR is set to be 70% in the experiments.

[1] https://sipi.usc.edu/database/database.php?volume=misc.

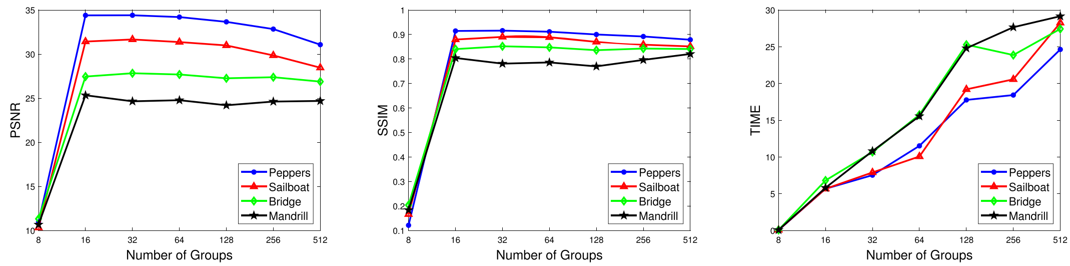

FIG. 2. The PSNR, SSIM and running time with different number of groups.

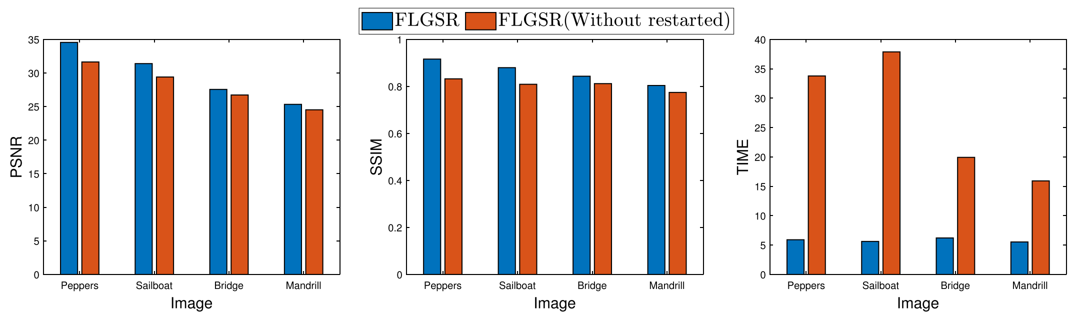

FIG. 3. The PSNR, SSIM and running time with different iteration methods for S.

5.1.1 *Effects of flexible grouping.* In this subsection, we compare the effects of different number of groups on our proposed algorithm FLGSR. The curves of PSNR, SSIM and running time with respect to different number of groups are shown in Fig. 2. From the recovery results, as the number of groups increases, the recovery effect of FLGSR first increases and then decreases, reaching the best near 16 groups. In terms of computational time, FLGSR takes significantly longer as the number of groups increases. When the number of groups reaches 512 (that is, each column is treated as a group like FGSR and GUIG), the time it consumes is about six times that of 16 groups. Therefore, the flexible grouping strategy of our proposed FLGSR method is effective in terms of both recovery quality and computational efficiency.

5.1.2 *Effects of the restarted technique.* In this subsection, we compare the effects of restarted technique on our proposed algorithm FLGSR. The bar charts of PSNR, SSIM and running time with respect to different iteration methods for S are shown in Fig. 3. From the PSNR and SSIM metrics, which measure the recovery effect, we can see that using the restarted technique on the Lagrange multiplier matrix (S) in our proposed algorithm FLGSR can enhance the recovery quality; from the running time, we can see that using the restarted technique on S can significantly reduce the computational time, which is about four times faster on average than not using the restarted technique on S.

TABLE 1 *Grayscale image inpainting performance comparison: PSNR, SSIM and running time*

Image	Index	FLGSR	FGSR	GUIG	NMF	SVT	ScaledGD	LRGeomCG
Peppers	PSNR	**34.391**	31.019	31.816	27.696	27.883	31.795	30.993
	SSIM	**0.905**	0.777	0.821	0.748	0.786	0.817	0.806
	Time	4.900	8.886	10.285	1.101	4.455	2.522	6.008
Sailboat	PSNR	**31.775**	29.850	29.170	24.724	26.004	28.864	28.932
	SSIM	**0.895**	0.811	0.781	0.673	0.776	0.778	0.780
	Time	5.295	9.598	11.335	0.339	3.773	2.383	3.313
Bridge	PSNR	**27.455**	26.169	24.372	23.179	20.977	24.344	23.999
	SSIM	**0.840**	0.778	0.725	0.603	0.634	0.718	0.715
	Time	5.233	9.815	10.808	0.403	2.363	2.476	12.312
Mandrill	PSNR	**25.301**	23.864	22.620	21.574	21.395	21.006	21.154
	SSIM	**0.802**	0.717	0.647	0.523	0.657	0.628	0.629
	Time	4.554	9.776	11.312	0.331	2.345	2.465	3.776

Bold values indicate the best results.

5.2 *Grayscale image inpainting*

In this subsection, we evaluate all the methods on the USC-SIPI Image Database.[2] For testing, we randomly select 20 images of size 512×512 pixels from this database. We set the SR to 70%.

We display the inpainting results of the four testing images (‘Peppers’, ‘Sailboat’, ‘Bridge’ and ‘Mandrill’) in Fig. 4. Enlarged views of parts of the recovered images clearly show the recovery differences. FLGSR recovers the details much better and preserves the surface of peppers, the sail on boat, the tree branch by bridge and the face of mandrill well. Table 1 provides the PSNR, SSIM and corresponding running times for each method, with the highest PSNR and SSIM values highlighted in bold. FLGSR consistently attains the highest PSNR and SSIM values among all methods. Notably, compared to the group sparsity-based methods FGSR and GUIG, FLGSR, not only achieves superior PSNR and SSIM values, but also requires less computation time. This improvement is attributed to our algorithm’s ability to leverage information from previously updated groups during the iteration of the current group in each cycle, enhancing convergence speed and accuracy, similar to the Gauss-Seidel approach’s advantages over Jacobi methods in linear system solutions.

In Fig. 5, we report the PSNR, SSIM and the algorithm running time of different methods on the 20 images. Our method achieves the best performance with an average improvement of 1.9 dB in PSNR and 0.07 in SSIM over the respective second best methods on each image, further verifying its advantages and robustness. In terms of time consumption, our method FLGSR is much faster than other compared methods based on group sparsity. In conclusion, it not only achieves the best inpainting results, but also runs very fast.

To further demonstrate the recovery performance, Fig. 6 presents the PSNR, SSIM and algorithm running time for different methods across varying SR. We can see that the proposed FLGSR method achieves superior accuracy in terms of PSNR and SSIM, except when $SR = 0.1$. Furthermore, as SR increases, the advantages of FLGSR become more pronounced in terms of PSNR and SSIM. Additionally, the running time of FLGSR remains around 4 seconds for different SR values, demonstrating greater stability and competitiveness compared to other algorithms.

[2] https://sipi.usc.edu/database/database.php?volume=misc.

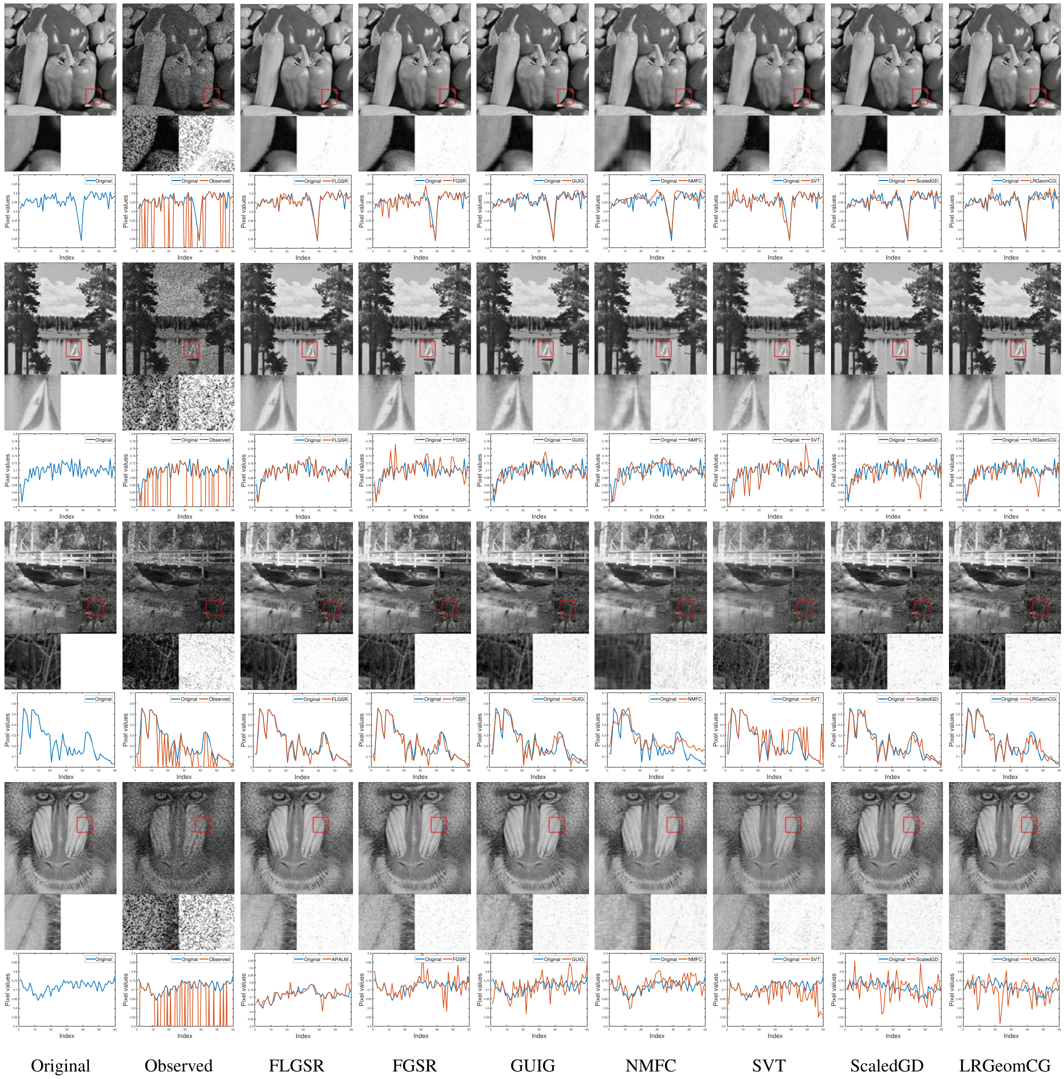

FIG. 4. Examples of grayscale image inpainting. From top to bottom, the plots respectively correspond to ‘Peppers’, ‘Sailboat’, ‘Bridge’ and ‘Mandrill’. For better visualization, we show the zoom-in region and the corresponding partial residuals of the region. Under each image, we show enlargements of a demarcated patch and the corresponding error map (difference from the Original). Error maps with less color information indicate better restoration performance.

5.3 *High altitude aerial image inpainting*

In this subsection, we test high altitude aerial (HAA) image set of size 512×512 pixels. The *SR* is set to 70%. Table 2 summarizes the PSNR, SSIM values and the corresponding running times of the compared algorithms. The highest PSNR and SSIM results are shown in bold. As observed, FLGSR consistently achieves the highest values in terms of all evaluation indexes, e.g., the proposed method achieves an approximately 1.66 dB gain in PSNR over the respective second best methods on each image. Figure 7

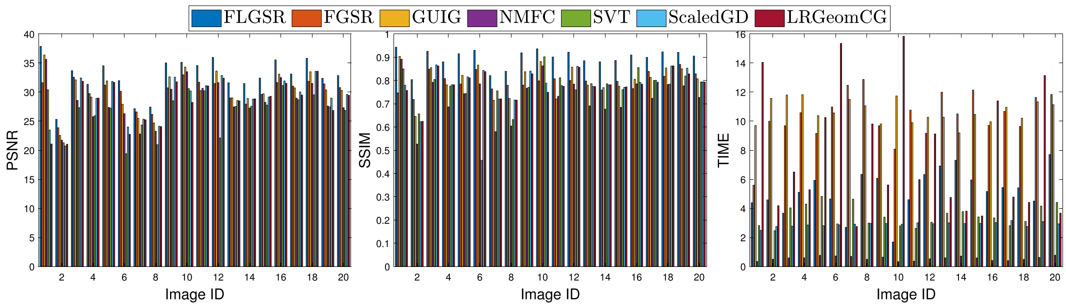

FIG. 5. Comparison of the PSNR, SSIM and the running time on the randomly selected 20 images.

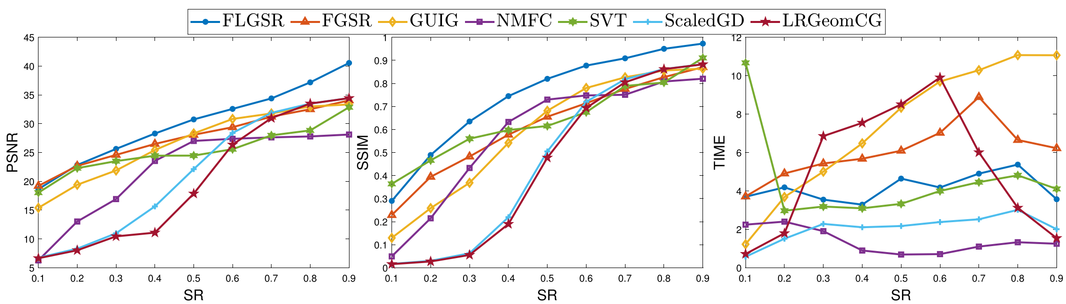

FIG. 6. Comparison of the PSNR, SSIM and the running time with different SR on the 'Peppers' image.

TABLE 2 *HAA image inpainting performance comparison: PSNR, SSIM and running time*

Image	Index	FLGSR	FGSR	GUIG	NMFC	SVT	ScaledGD	LRGeomCG
San Diego	PSNR	**29.843**	28.182	27.113	22.539	25.133	27.029	27.066
	SSIM	**0.866**	0.794	0.743	0.585	0.768	0.746	0.746
	Time	7.578	10.651	17.139	0.358	4.356	2.559	2.803
Woodland Hills	PSNR	**28.819**	26.873	26.143	24.330	23.287	25.932	25.993
	SSIM	**0.857**	0.785	0.730	0.591	0.682	0.730	0.731
	Time	6.521	10.159	11.581	0.387	2.555	2.560	5.404

Bold values indicate the best results.

shows a visualized comparison of the recovery images. As observed, FLGSR produces the best visual results. In addition, it can be seen that NMF and SVT still contain a certain amount of noise. The high altitude aerial image inpainting results are also consistent with the grayscale image inpainting results and all these demonstrate that our FLGSR results are much better than other methods, both in visual quality and in terms of PSNR, and SSIM.

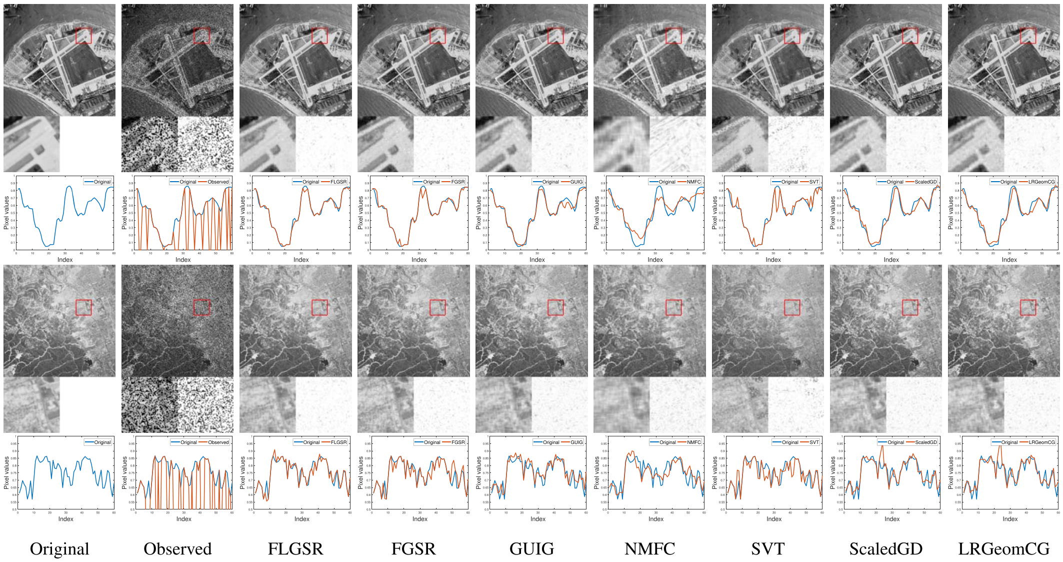

FIG. 7. Examples of HAA image inpainting. From top to bottom, the plots respectively correspond to 'San Diego' and 'Woodland Hills'. For better visualization, we show the zoom-in region and the corresponding partial residuals of the region. Under each image, we show enlargements of a demarcated patch and the corresponding error map (difference from the Original). Error maps with less color information indicate better restoration performance.

6. Conclusions and future work

In this paper, we have proposed a new group sparsity approach to the LRMR problem, which can recover low rank matrices from incomplete observations. Specifically, we have introduced the FLGSR, a novel regularizer that can group multiple columns of a matrix as a unit based on the data structure. By doing so, we have proved the equivalence between the matrix rank and the FLGSR, and shown that the LRMR problem with either of them has the same global minimizers. Furthermore, we have also established the equivalence between the relaxed and the penalty formulations of the LRMR problem with FLGSR. To optimize this model, we have devised an efficient algorithm to solve the LRMR problem with FLGSR, and analysed its convergence properties. Finally, we have demonstrated the superiority of our method over state-of-the-art methods in terms of recovery accuracy, visual quality and computational efficiency on both grayscale images and high altitude aerial images.

The FGSR has been generalized to tensor completion Fan *et al.* (2023), indicating the potential for extending the proposed FLGSR to tensor recovery problems. This future research direction could explore the theoretical underpinnings and practical applications of FLGSR in higher dimensional data structures. Additionally, incorporating adaptive mechanisms into the FLGSR framework to dynamically adjust regularization parameters based on data characteristics could enhance the method's flexibility and robustness in handling diverse datasets.

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Conflicts of interest

All authors declare that there are no conflicts of interest in this paper.

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